



Chapter 10 Torque and Rotational Dynamics

“Horsepower sells cars, torque wins races.”

—Carroll Shelby

So far, we’ve explored how objects move from one place to another. We first explored how objects moved in one dimension, then we moved on to two dimensions. We described objects moving in a line, a parabola, and even a circle. We’re still missing something, though. Motion doesn’t always involve going from one place to another. You could spin around in a chair, for instance. So to complete our description of motion, we need to explore rotation: how to describe rotational motion and how to create it with torque.

ROTATIONAL MOTION

Previously, we covered objects that undergo circular motion. This chapter focuses on taking those objects and spinning them. Previous equations involved objects moving in a linear orientation or being manipulated into circular orbit. With rotational motion, we will need to take on a new set of equations that are analogous to the physics of linear motion.

If we recall from before, an object's mass measures its inertia—its resistance to acceleration. The greater the inertia on an object, the harder it is to change its velocity. “Harder to change its velocity” means the object is harder to deliver an acceleration on—which in turn means the greater the inertia, the greater the force that is required in order for an object to be moved. Comparing two objects, if Object 1 has greater inertia than Object 2 and the same force is applied on both objects, Object 1 will undergo a smaller acceleration.

In the linear model, we put these in terms of force, mass, acceleration, and velocity. When it comes to rotational kinematics, we need to change up a few of these terms:

Linear Kinematics	Rotational Kinematics
Force	Torque (τ)
Mass	Moment of Inertia (I)
Acceleration	Angular Acceleration (α)
$F_{\text{net}} = ma$	$\tau_{\text{net}} = I\alpha$
Velocity	Angular Velocity (ω)

Rotational Kinematics

Like our linear equations, which are used to determine the distance (x), velocity (v), and linear acceleration, we use rotational equations to determine the same factors. In this section, we will go over angular distance (θ), angular velocity (ω), and angular acceleration (α). Finally, we will explore the relationship between these three rotational parameters and the linear parameters.

It's All Greek to Me

If you haven't seen these symbols before, here's a quick pronunciation guide:

θ = theta

ω = omega

α = alpha

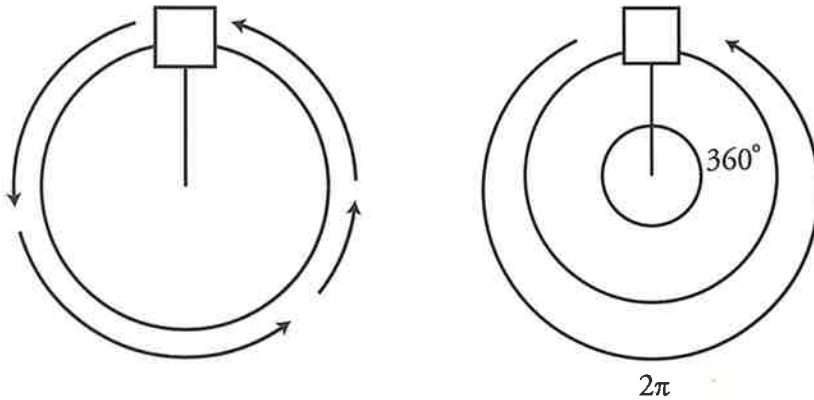
As you work your way through math and physics, you'll eventually learn almost all the Greek letters.

Let's start with some basic definitions.

What is **angular displacement**?

What is **translational displacement**?

If you look at a circle, you can see that one time around the circle (1 revolution) equals 2π radians, or 360° .



The linear position or physical distance traveled around the circle (Δs) can be related to angular position and the radius of the circle (r) via this equation:

$$\theta = \frac{s}{r}$$

What Is Angular Velocity ($\vec{\omega}$)?

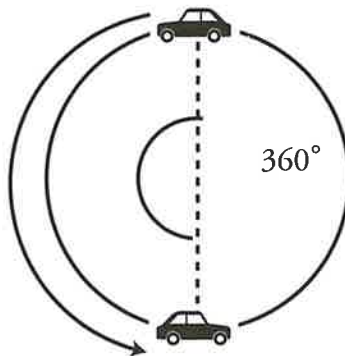
Angular Velocity	Linear Velocity
$\vec{\omega} = \frac{\Delta\theta}{\Delta t}$	$\vec{V} = \frac{\Delta s}{\Delta t}$
Units = $\frac{\text{Rad}}{\text{s}}$ or $\frac{\text{Rev}}{\text{min}}$	Units = m/s
The angular velocity equals change in angular displacement divided by change in time.	The linear velocity equals change in distance (Δx) divided by change in time.

We can relate angular velocity to linear velocity via the following:

$$\vec{V} = r\vec{\omega}$$

Note, just like \vec{V} , $\vec{\omega}$ has direction!

The **Right-Hand Rule** states that you must wrap your fingers around the object's path. Let's take the example of a toy car going around a circle.



If you follow the car's path, you will find your fingers are wrapping counterclockwise. Your thumb is the direction for angular velocity. In this case, it points off of the page. In physics, we write the direction like this \odot . If it were on the page, we would use this \otimes .

What Is Angular Acceleration (α)?

Angular acceleration, $\alpha = \frac{\Delta\omega}{\Delta t}$

Units: radians/ s^2

Many of the rotational kinematics equations reflect linear kinematics equations.

	Rotational Motion	Linear Motion
Big Five #1:	$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t$	$\Delta x = \frac{1}{2}(v_0 + v)t$
Big Five #2:	$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
Big Five #3:	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x = x_0 + v_0 t + \frac{1}{2}at^2$
Big Five #4:	$\theta = \theta_0 + \omega t - \frac{1}{2}\alpha t^2$	$x = x_0 + vt - \frac{1}{2}at^2$
Big Five #5:	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	$v^2 = v_0^2 + 2a(x - x_0)$

Example 1 Four children climb on a carousel that is initially at rest. If the carousel accelerates to 0.4 radians per second within 10 seconds, what is the angular acceleration? What is its linear rotation 3 m from the axis of rotation?

Solution.

$$\omega_i = 0$$

$$\omega_0 = 0.4 \text{ rad/s}$$

$$\alpha = \frac{\omega_0 - \omega_i}{t} = \frac{0.4 - 0}{10} = 0.04 \text{ rad/s}^2$$

$$\Delta\theta$$

$$\alpha = ?$$

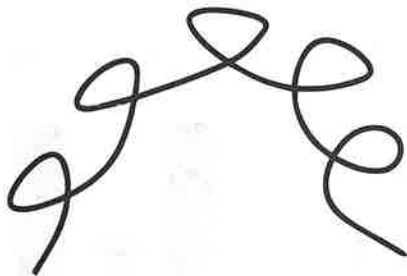
$$T = 10 \text{ sec}$$

To find linear acceleration: $a = r\alpha = (3 \text{ m})(0.04 \text{ rad/s}^2) = 0.12 \text{ m/s}^2$

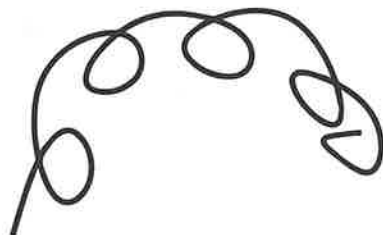
CENTER OF MASS

In the preceding chapters, objects were treated as though they were each a single particle. In many force-diagrams, we have said that all the force is being delivered at a single point on the object. What makes this point the center of mass? And why do we associate all the force being delivered at this single point on the object?

Imagine a series of experiments. We walk into a large room with a hammer and a small light that we can attach to the hammer. In the first experiment, we hold the hammer and we attach a light to the very end of the hammer. Then, we turn off the light and throw the hammer across the room. If we trace the path of the hammer, we notice that it gives a weird spiral-shaped path as follows:



Then we repeat the experiment. This time, let's attach the small light to the head of the hammer. Once again we turn off the light, throw the hammer across the room, and trace the path of the hammer. This time we notice that it follows another spiral-shaped path:



You may have heard some people use the term "center of gravity" instead of "center of mass." While there is a technical difference between them, it's not important for this test. The two can be used interchangeably for any AP Physics 1 discussion.

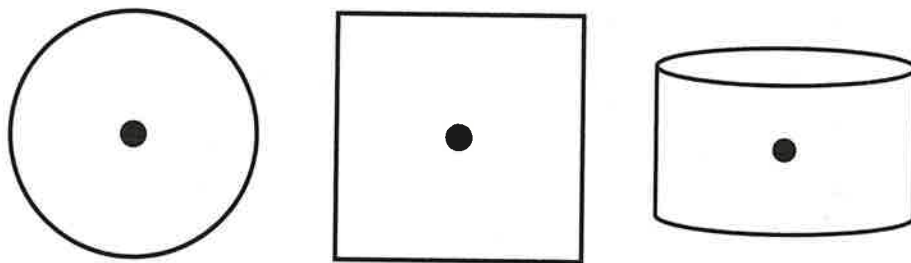
If we keep doing the experiment, after countless trials, you will notice that at a specific point, the hammer makes a parabolic path seen in the following drawing:



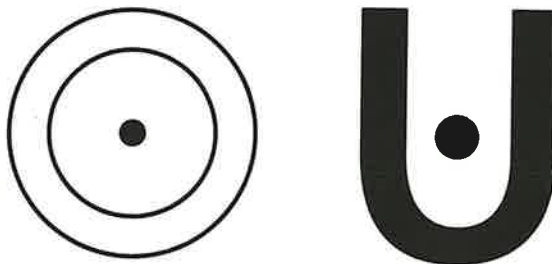
Apparently, there was something important about that specific point. All the other points gave spiraled trajectories, but this one gave a smooth parabolic path. Upon further investigation, if we place that point on our fingers, we notice that the hammer balances nicely and is perfectly horizontal with the floor.

This certain point is the center of mass. Another way of looking at it is to say that the center of mass is the point at which we could consider all the mass of the object to be concentrated.

For a simple object such as a sphere, block, or cylinder, whose density is constant (a term in physics we call *homogeneous*), the center of mass is at its geometric center.



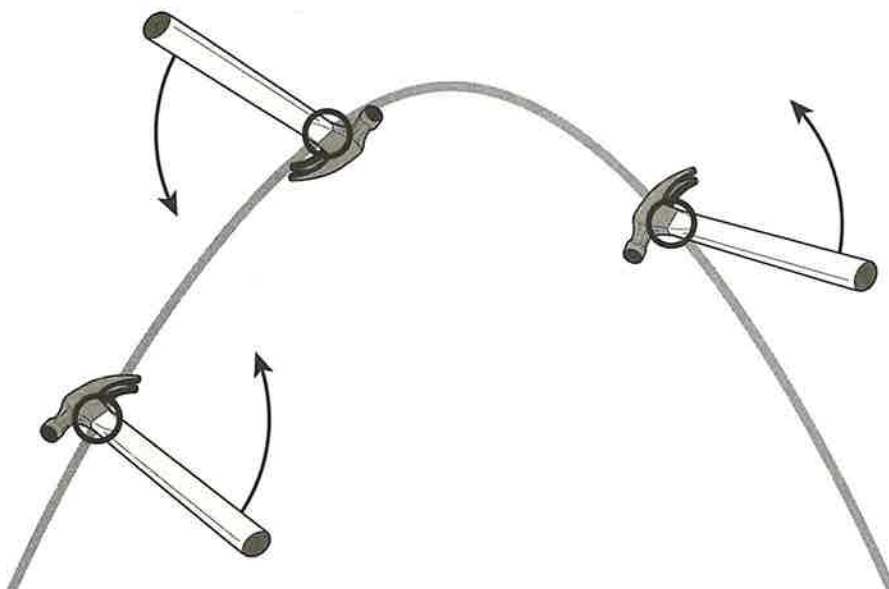
In some cases, the center of mass is not located on the body of the object:



While equations for the location of the center of mass exist, you don't need to know them for the AP Exam. Only a qualitative understanding is required.

For objects that are irregularly shaped or have different densities throughout, like a hammer, the center of mass is located closer to where most of the mass resides.

Thus, a point close to the head of the hammer would follow a parabolic path when we throw the hammer.



Heads Up!
Somebody's throwing hammers!

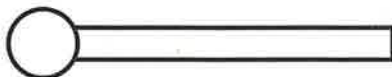
TORQUE

We can tie a ball to a string and make it undergo circular motion, but how would we make that ball itself spin? We could simply palm the ball and rotate our hand, or we could put our hands on opposite sides of the ball and push one hand forward and the other backward. In both cases, in order to make an object's center of mass accelerate, we need to exert a force. In order to make an object spin, we need to exert a torque.

Torque is the measure of a force's effectiveness at making an object spin or rotate. (More precisely, it's the measure of a force's effectiveness at making an object accelerate rotationally.) If an object is initially at rest and then it starts to spin, something must have exerted a torque. And if an object is already spinning, something would have to exert a torque to get it to stop spinning.

All systems that can spin or rotate have a "center" of turning. This is the point that does not move while the remainder of the object is rotating, effectively becoming the center of the circle. There are many terms used to describe this point, including pivot point and fulcrum.

Torque has always been a topic that students have difficulty understanding. So let's go through a few examples and ask a few questions before we delve into torque:



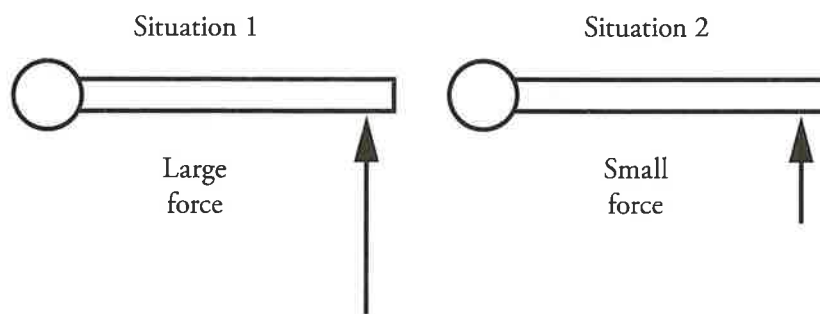
The drawing above shows a door with its hinge (pivot point) located on the left side of the door. You can try some of these examples at home on a door in your house to get a better understanding. Let's pose two different situations for trying to close the door:

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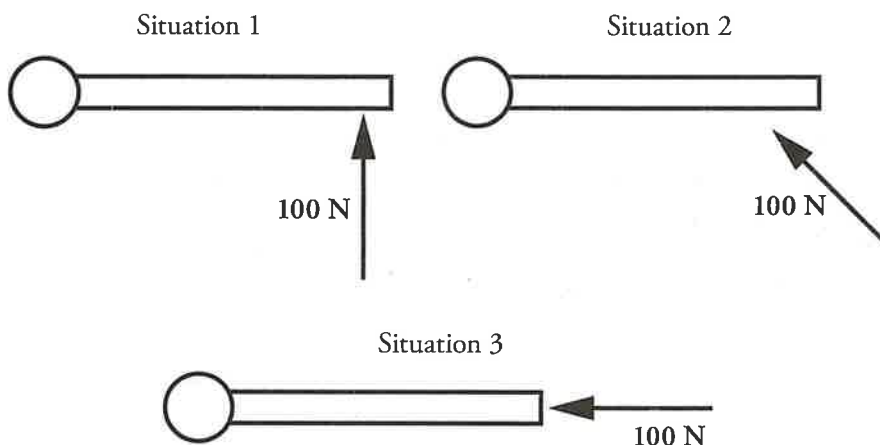


SCENARIO 1



In Situation 1, the door will close the fastest because of the greater force used. Now in the same example, let's say that a 100 N force is going to be applied to the door but at different angles:

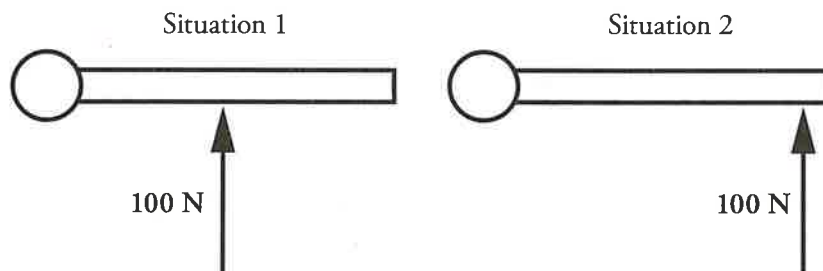
SCENARIO 2



The door will close the fastest in Situation 1 and the second fastest in Situation 2. Situation 3 involves merely pushing on the door to no avail; the door will not close.

Now let's use the same door example and apply the force to different parts of the door:

SCENARIO 3



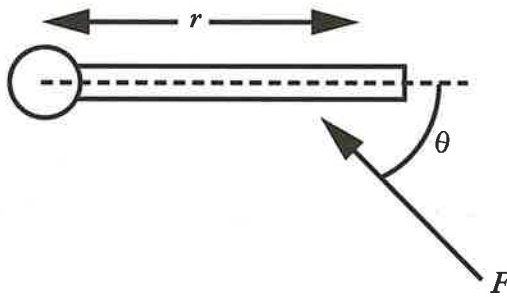
If you try this at home, you will notice that if you push the door as in Situation 2, it will be easier to close the door than if you tried to push the door as in Situation 1.

If you noticed in the three scenarios, there were a few points that mattered when trying to close the door. In Scenario 1, the amount of force used to close the door mattered (magnitude of force). In Scenario 2, the angle in which we pushed the door mattered (angle). And in Scenario 3, the place in which we pushed mattered (radius).

Our force's effectiveness at making something spin or rotate was determined by three factors:

1. the magnitude of force (F)
2. the angle (θ)
3. the radius (r)

These three factors give us our torque equation:



$$\tau = rF \sin\theta$$

There's no special name for this unit (torque): it's just called a newton-meter. Because it is not in newtons, torque is NOT a force. (Torque is a measure of how much a force acting on an object causes that object to rotate.) In Scenario 2, Situation 3, a force was being applied straight on to the door directed straight into the pivot point. The magnitude of force did not suddenly disappear, but it was not effective at closing the door. Torque is the rotational equivalent of force in trying to make something accelerate rotationally.

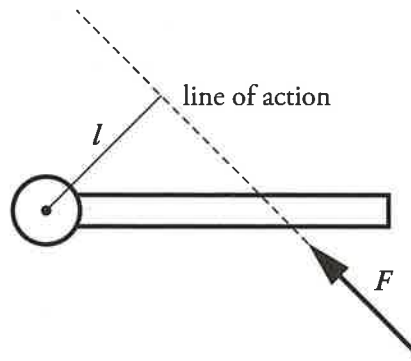
Another way to calculate torque is using the lever arm. For this method, we consider the line of action of the force, which is the line on which the force vector sits. The lever arm is the perpendicular distance from the pivot to the line of action of the force.

Why Sine?

In math, this is known as a cross product between your force and your radius. It is written as $\tau = (r \times F)$.

Don't Confuse the Units

In the previous chapter, a newton-meter became a joule. This is not the case with torque.



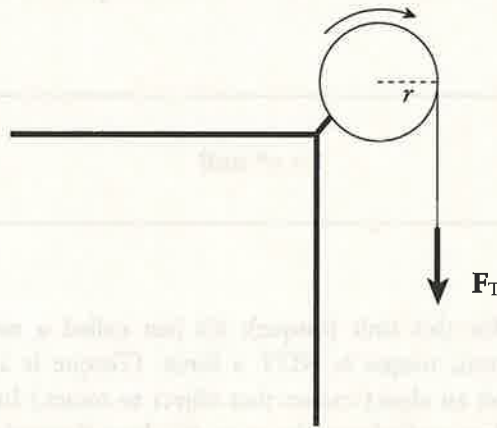
Using the lever arm, torque is defined as:

$$\tau = lF$$

If we compare the two methods, we notice that $l = r \sin \theta$, so both methods give the same answer for torque, as we'd expect. Generally, only use the lever arm equation when the lever arm is fairly easy to identify in a problem.

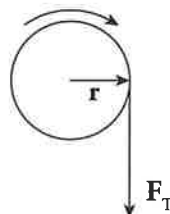
Torque problems usually involve putting systems in equilibrium.

Example 2 A student pulls down with a force of 40 N on a rope that winds around a pulley of radius 5 cm.



What's the torque of this force?

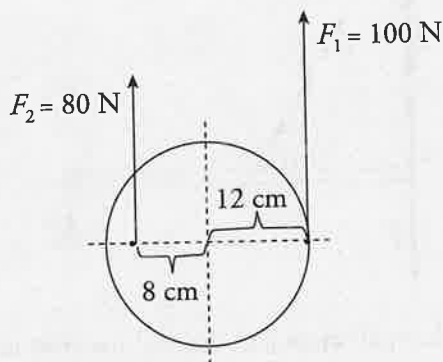
Solution. Since the tension force, F_T , is tangent to the pulley, it is perpendicular to the radius vector r at the point of contact:



Therefore, the torque produced by this tension force is simply:

$$\tau = rF_{\perp} = (0.05 \text{ m})(40 \text{ N}) = 2 \text{ N}\cdot\text{m}$$

Example 3 What is the net torque on the cylinder shown below, which is pinned at the center?



Solution. Each of the two forces produces a torque, but these torques oppose each other. The torque of F_1 is counterclockwise, and the torque of F_2 is clockwise. This can be visualized by imagining the effect of each force, assuming that the other was absent.

The **net torque** is the sum of all the torques. Counting a counterclockwise torque as positive and a clockwise torque as negative, we have:

$$\tau_1 = +r_1F_1 = +(0.12 \text{ m})(100 \text{ N}) = +12 \text{ N}\cdot\text{m}$$

and

$$\tau_2 = -r_2F_2 = -(0.08 \text{ m})(80 \text{ N}) = -6.4 \text{ N}\cdot\text{m}$$

so

$$\tau_{\text{net}} = \Sigma\tau = \tau_1 + \tau_2 = (+12 \text{ N}\cdot\text{m}) + (-6.4 \text{ N}\cdot\text{m}) = +5.6 \text{ N}\cdot\text{m}$$

EQUILIBRIUM

An object is said to be in **translational equilibrium** if the sum of the forces acting on it is zero—that is, if $F_{\text{net}} = 0$. Similarly, an object is said to be in **rotational equilibrium** if the sum of the torques acting on it is zero—that is, if $\tau_{\text{net}} = 0$. The term *equilibrium* by itself means both translational and rotational equilibrium. A body in equilibrium may be in motion; $F_{\text{net}} = 0$ does not mean that the velocity is zero; it only means that the velocity is constant. Similarly, $\tau_{\text{net}} = 0$ does not mean that the angular velocity is zero; it only means that it's constant. If an object is at rest, then it is said to be in **static equilibrium**.

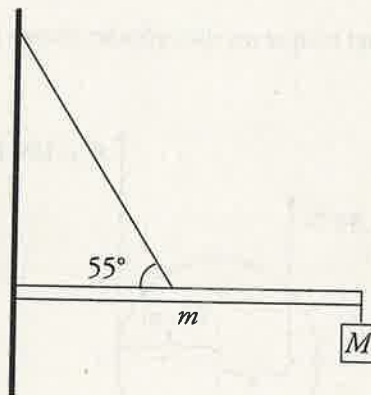
Keep Orientation in Mind

With spinning objects, we will often be referring to directions as clockwise or counterclockwise.

It's All About Balance

Equilibrium problems are all about balancing one force's effectiveness at turning something clockwise with another force's effectiveness at turning something counterclockwise.

Example 4 A uniform bar of mass m and length L extends horizontally from a wall. A supporting wire connects the wall to the bar's midpoint, making an angle of 55° with the bar. A sign of mass M hangs from the end of the bar.

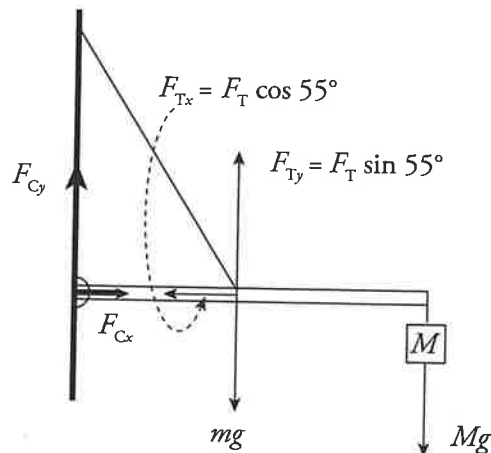


If the system is in static equilibrium and the wall has friction, determine the tension in the wire and the strength of the force exerted on the bar by the wall if $m = 8 \text{ kg}$ and $M = 12 \text{ kg}$.

Choose Your Strategy

Some problems are easier when using the center of mass, whereas some problems are easier using torque. Choose which one is easier for you.

Solution. Let F_C denote the (contact) force exerted by the wall on the bar. In order to simplify our work, we can write F_C in terms of its horizontal component, F_{Cx} , and its vertical component, F_{Cy} . Also, if F_T is the tension in the wire, then $F_{Tx} = F_T \cos 55^\circ$ and $F_{Ty} = F_T \sin 55^\circ$ are its components. This gives us the following force diagram:



The first condition for equilibrium requires that the sum of the horizontal forces is zero and the sum of the vertical forces is zero:

$$\Sigma F_x = 0: \quad F_{Cx} - F_T \cos 55^\circ = 0 \quad (1)$$

$$\Sigma F_y = 0: \quad F_{Cy} + F_T \sin 55^\circ - mg - Mg = 0 \quad (2)$$

We notice immediately that we have more unknowns (F_{Cx} , F_{Cy} , F_T) than equations, so this system cannot be solved as is. The second condition for equilibrium requires that the sum of the torques about any point is equal to zero. Choosing the contact point between the bar and the wall as our pivot, only three of the forces in the diagram above produce torque: F_{Ty} produces a counterclockwise torque, both mg and Mg produce clockwise torques, and the sum of the three torques must equal zero. From the definition $\tau = rF \sin \theta$, and taking counterclockwise torque as positive and clockwise torque as negative, we have:

$$\Sigma \tau = 0: \quad \frac{1}{2} \lambda F_{Ty} - \frac{1}{2} mg \lambda - LMg \lambda = 0 \quad (3)$$

Note that you can divide out the r term here, so r will not factor into the final equation. The above equation contains only one unknown and can be solved immediately:

$$\begin{aligned} \frac{1}{2} F_{Ty} &= \frac{1}{2} mg + LMg \\ F_{Ty} &= mg + 2Mg = (m + 2M)g \end{aligned}$$

Since $F_{Ty} = F_T \sin 55^\circ$, we can find that:

$$\begin{aligned} F_T \sin 55^\circ &= (m + 2M)g \quad \Rightarrow \quad F_T = \frac{(m + 2M)g}{\sin 55^\circ} \\ &= \frac{(8 + 2 \cdot 12)(10)}{\sin 55^\circ} \\ &= 390 \text{ N} \end{aligned}$$

Substituting this result into Equation (1) gives us F_{Cx} :

$$F_{Cx} = F_T \cos 55^\circ = \frac{(m + 2M)g}{\sin 55^\circ} \cos 55^\circ = (8 + 2 \cdot 12)(10) \cot 55^\circ = 220 \text{ N}$$

$$\text{Note: We use } \cot 55^\circ \text{ here, because } \cot 55^\circ = \frac{\cos 55^\circ}{\sin 55^\circ}.$$

And finally, from Equation (2), we get:

$$\begin{aligned} F_{Cy} &= mg + Mg - F_T \sin 55^\circ \\ &= mg + Mg - \frac{(m + 2M)g}{\sin 55^\circ} \sin 55^\circ \\ &= -Mg \\ &= -(12)(10) \\ &= -120 \text{ N} \end{aligned}$$

The fact that F_{Cy} turned out to be negative simply means that in our original force diagram, the vector F_{Cy} points in the direction opposite to how we drew it. That is, F_{Cy} points downward. Therefore, the magnitude of the total force exerted by the wall on the bar is:

$$F_C = \sqrt{(F_{Cx})^2 + (F_{Cy})^2} = \sqrt{220^2 + 120^2} = 250 \text{ N}$$

ROTATIONAL INERTIA

Now that we've studied torque and rotation, we can finally put together the pieces of making an object spin. An object's **rotational inertia** (also known as the moment of inertia) is defined by how difficult it is to change the angular velocity of an object. Think of mass as translational inertia since it measures an object's resistance to translational acceleration, given by a in $F = ma$. Then, just as translational inertia, m , tells us how resistant an object is to translation acceleration, an object's rotational inertia, I , tells us how resistant the object is to rotational acceleration.

Rotational acceleration or angular acceleration is the same as translational acceleration except we are taking an object changing its angular velocity, or rate of rotation. In order to achieve this acceleration, a force is required. In terms of rotational inertia, this force creates a torque that determines the angular acceleration. This leads to the rotational kinematics version of Newton's Second Law:

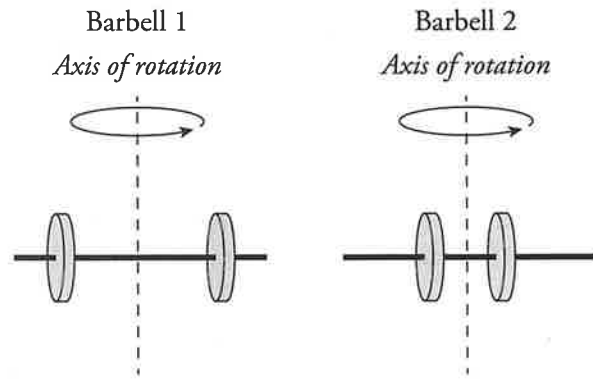
$$\tau_{\text{net}} = I\alpha$$

or

$$\alpha_{\text{sys}} = \frac{\sum \tau}{I_{\text{sys}}} = \frac{\tau_{\text{net}}}{I_{\text{sys}}}$$

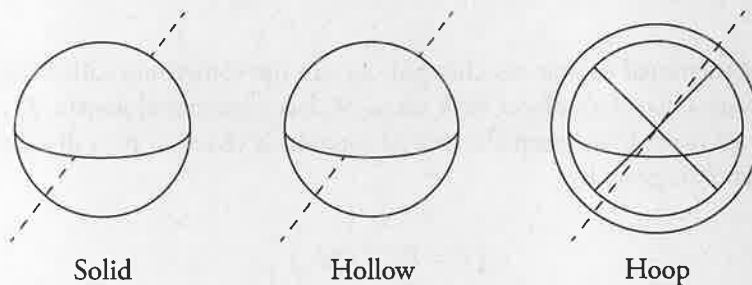
These equations make some key relationships. The larger the rotational inertia (moment of inertia) is, the smaller the value of the angular acceleration, α , will be for a given net torque on an object. If Object 1 has a greater rotational inertia than Object 2, then it will be more difficult to change the rotation of Object 1 compared to Object 2. More precisely, a greater torque would be required to give Object 1 the same rotational acceleration as Object 2.

So, how do we find the rotational inertia of an object? It depends on the object's mass, but there is more to it than that. Two objects can have the same mass but different rotational inertias. Imagine a barbell with a weight near each end and an identical barbell with the weights pushed near the middle of the bar. These two barbells have the same mass, but their rotational inertias are different. Specifically, if we wanted to rotate each bar around its midpoint, we would find it more difficult to start spinning the barbell with the weights at the ends. This indicates that, because the masses are farther away from the axis of rotation, the first barbell has a greater value of I .



So rotational inertia is dependent on two things: how much mass the object has and how far away the mass is from the axis of rotation.

Example 5 A solid sphere, a hollow sphere, and a hoop all have the same mass and radii. Each are suspended on different axes passing through their center as shown below and an identical torque is applied to each. Rank the magnitude of angular acceleration each object feels from greatest to least.



Solution. To rank the angular acceleration, we need to consider how the rotational inertias compare for each object. In the case of a solid sphere, there is quite a bit of mass very close to the axis of rotation, which implies a small rotational inertia. In the case of a hollow sphere, there is still some mass near the axis of rotation, but not as much, which implies a larger rotational inertia. Finally, in the case of the hoop, all of the mass is a good distance from the axis of rotation, which implies a very large rotational inertia. Therefore,

$$I_{\text{solid}} < I_{\text{hollow}} < I_{\text{hoop}}$$

The larger the rotational inertia, the smaller the angular acceleration, so

$$\alpha_{\text{solid}} > \alpha_{\text{hollow}} > \alpha_{\text{hoop}}$$

Finding the exact values of I for different objects is actually quite complex and is not something you'll need to know for the exam. However, there is a simple equation for finding the rotational inertia from a point mass, m , a distance of r from the axis of rotation.

$$(I = mr^2)$$

The AP Exam will give you the moment of inertia for any objects you need besides that of a simple point mass.

Furthermore, if we have multiple objects rotating about a common axis of rotation. The total rotational inertia is simply the sum of the rotational inertia of each object.

$$(I = \sum m_i r_i^2)$$

PARALLEL AXIS THEROEM

So far, we've only looked at an axis of rotation that passes through the center of mass of an object. However, this doesn't always need to be the case. Image a solid wooden rod. We could drill a hole in the center and rotate around an axis there, or we could attach a hinge to the edge of the rod and have it rotate around an axis there. Changing the axis of rotation like this will change the rotational inertia of the object, since more of the mass is farther from the axis of rotation.

To find how the rotational inertia has changed, we can use something called the **parallel axis theorem**. This states that if an object with mass, M , has a rotational inertia, I_{cm} , when rotated about its center of mass, then when the axis of rotation is changed by a distance, d , the new rotational inertia, I' , is given by

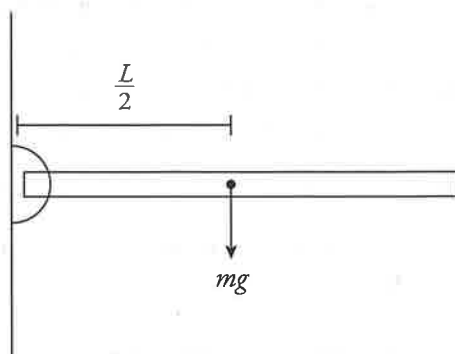
$$(I' = I_{cm} + Md^2)$$

Example 6 A uniform bar of mass m and length L has a moment of inertia of $\frac{1}{12}mL^2$ when rotated about its center of mass. A hinge is used to attach a uniform 10 kg bar with a length of 2 m to a wall. The bar is then held horizontally and released. What is the angular acceleration of the bar about its hinge immediately after it has been released?

Solution. First, we'll need to find the rotational inertia about the end of the bar. Changing the axis from the center to the end means the axis has changed by a distance of $L/2$, so using the Parallel Axis Theorem:

$$I' = I_{cm} + Md^2 = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{12}mL^2 + \frac{1}{4}mL^2 = \frac{1}{3}mL^2$$

Next, consider the diagram below:



When held horizontal, the force of gravity creates a torque on the bar given by $\frac{L}{2}mg$. Therefore, using the rotational form of Newton's Second Law yields

$$\alpha = \frac{\tau}{I} = \frac{\frac{L}{2}mg}{\frac{1}{3}mL^2} = \frac{3g}{2L} = \frac{3(10)}{2(2)} = 7.5 \text{ rad/s}^2$$

ANGULAR MOMENTUM

The same way that objects moving in a straight line have linear momentum, objects experiencing rotation have angular momentum. **Angular momentum** is a vector quantity denoted by \mathbf{L} . Similar to the other rotational vectors we've covered so far, angular momentum has only two possible directions: clockwise and counterclockwise. Furthermore, everything you learned about linear momentum back in Chapter 8 still applies. All you have to do is replace the linear terms in the equations you already know with their analogous rotational terms.

For example, linear momentum is defined as the product of mass and velocity, and angular momentum is given by the following equation:

$$\mathbf{L} = I\boldsymbol{\omega}$$

which shows *angular momentum* is the product of the rotational inertia and angular velocity.

Additionally, we also have a rotational version of the Impulse–Momentum Theorem, which states that the change in linear momentum is the product of force and the change in time. A similar equation exists for finding the change in angular momentum:

$$\Delta\mathbf{L} = \boldsymbol{\tau}\Delta t$$

Just like how force causes linear acceleration, which leads to a change in linear momentum, torque causes angular acceleration which leads to a change in angular momentum.

Example 7 A spherical object on an axis of rotation through its center has a rotational inertia of $10 \text{ kg} \cdot \text{m}^2$ and a radius of 50 cm. It starts at rest when a force of 50 N is applied at the edge of the object and parallel to its surface. After some time, t , the angular velocity of the object is now 30 rad/s. What is t ?

Solution. We could solve this using the rotational version of Newton's Second Law, but it's a bit faster to use impulse and momentum:

$$\Delta \mathbf{L} = \tau \Delta t = rFt = I\omega_f - I\omega_i, \text{ so}$$

$$t = \frac{I\omega_f - I\omega_i}{rF} = \frac{(10)(30) - 0}{(0.5)(50)} = 12 \text{ s}$$

In addition to the above equations, it's also important to know that, just like linear momentum, angular momentum is conserved for any isolated system.

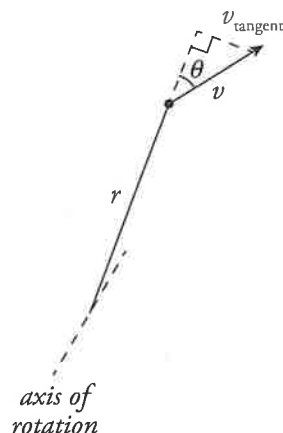
Example 8 A man is standing halfway between the center and one edge of a rotating plank. If he wants to slow down the rotational speed, should he walk toward the center or the edge?

Solution. Because this is an isolated system, we know angular momentum will be conserved. Since $\mathbf{L} = I\omega$, that means our desired decrease to rotational velocity can be achieved by increasing the rotational inertia. Earlier, we learned that rotational inertia is greater when the mass is farther from the axis of rotation. In this case, that means the man should walk toward the edge.

Rotating objects have an angular momentum based on their angular velocity and rotational inertia, but what about objects that aren't rotating about the axis? What if we have objects traveling in a straight line? It turns out that they too contribute to the angular momentum of a rotating system. To see how, remember that a point mass has a rotational inertia of mr^2 and that the tangential velocity of an object is given by $v_{\text{tangential}} = r\omega$. If a point mass is traveling past an axis of rotation in a straight line with a velocity, v , then the angular momentum of that object would be

$$\begin{aligned} \mathbf{L} &= I\omega \\ &= mr^2\omega \\ &= mr(r\omega) \\ &= mrv_{\text{tangential}} \\ &= mrv \sin \theta \end{aligned}$$

where θ is the angle between a line from the axis of rotation and the velocity vector.

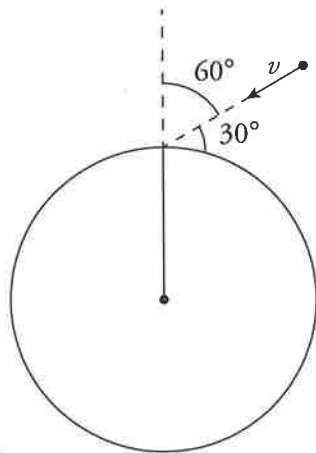


So for an object traveling in a straight line past an axis of rotation, its angular momentum is given by

$$(L = mrv \sin \theta)$$

Example 9 A disk with a rotational inertia of $40 \text{ kg} \cdot \text{m}^2$ and radius of 2 m is rotating with an angular velocity of 20 rad/s counterclockwise. A clay ball with a mass of 1 kg traveling at a speed of 15 m/s to the left strikes the top outside edge of the disk making a 30° angle with its surface and sticks to the disk. What is the angular velocity of the disk-ball system after the ball has struck the disk?

Solution. Because the disk-ball system is an isolated system, we know angular momentum will be conserved. The total angular momentum is the sum of the angular momentum of the disk and the angular momentum of the ball. To find the initial angular momentum of the ball, we need the angle the ball makes with the line from the axis of rotation. Consider the following diagram:



From this we can see the angle between the line from the axis and the velocity at the moment of impact will be 60° . Since this strike is in the same direction as the counterclockwise rotation, the initial angular momentum of the ball should be positive. Furthermore, once the ball strikes the disk, it will rotate with the same angular velocity of the disk. Therefore, setting the initial and final angular momentums equal yields

$$L_i = L_f$$

$$I_{\text{disk}}\omega_i + mvr \sin\theta = I_{\text{disk}}\omega_f + I_{\text{ball}}\omega_f$$

Using $I_{\text{ball}} = mr^2$ gives

$$\omega_f = \frac{I_{\text{disk}}\omega_i + mvr \sin\theta}{I_{\text{disk}} + mr^2} = \frac{(40)(20) + (1)(15)(2)\sin(60^\circ)}{40 + (1)(2)^2} \approx 18.8 \text{ rad/s}$$

ROTATIONAL KINETIC ENERGY

Kinetic energy is the energy of motion. Since motion can be translational or rotational, kinetic energy can consist of translational and rotational kinetic energy contributions. Similar to other rotational quantities, **rotational kinetic energy** is calculated by replacing the terms in the equation for translational kinetic energy with their rotational analogs. Therefore, rotational kinetic energy is given by the following equation:

$$\left(K_r = \frac{1}{2} I\omega^2 \right)$$

Rotational kinetic energy is just another form of kinetic energy. Therefore, when considering concepts like conservation of total mechanical energy, kinetic energy can be broken down into translational and rotational terms.

$$K_i + U_i = K_f + U_f$$

$$(K_{t,i} + K_{r,i}) + U_i = (K_{t,f} + K_{r,f}) + U_f$$

Example 10 A 0.1 kg yo-yo with a rotational inertia of $5 \times 10^{-5} \text{ kg} \cdot \text{m}^2$ attached to a string of length 30 cm is released from rest. Just before it reaches the bottom of the string, it is moving with a speed of 0.75 m/s. What is the magnitude of the angular velocity in revolutions per second of the yo-yo just before it reaches the bottom of the string?

Solution. When the yo-yo is falling, its gravitational potential energy is being converted into both translational and rotational kinetic energy, and the Conservation of Energy equation becomes

$$K_i + U_i = K_f + U_f$$

$$0 + mgh = \left(\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\right) + 0$$

Therefore, the angular velocity is

$$\omega = \sqrt{\frac{2mgh - mv^2}{I}} = \sqrt{\frac{2(0.2)(10)(0.3) - (0.2)(0.75)^2}{5 \times 10^{-5}}} = 147.5 \text{ rad/s} = 23.5 \text{ rev/s}$$

One special scenario is when an object with a circular cross-section rolls without slipping. In this case, the object's angular speed is related to its translational speed and its radius, $\omega = v/r$. Therefore, the object's rotational kinetic energy is proportional to its translational kinetic energy.

Example 11 A ball with a rotational inertia of $\frac{2}{3}MR^2$ rolls without slipping with a constant speed v . What fraction of its kinetic energy is rotational?

Solution. For a ball that rolls without slipping, the rotational kinetic energy of the ball is:

$$K_r = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{2}{3}MR^2\right)\left(\frac{v}{R}\right)^2 = \frac{2}{3}\left(\frac{1}{2}Mv^2\right) = \frac{2}{3}K_t$$

Therefore, the fraction of its total kinetic energy that is rotational is:

$$\frac{K_r}{K_r + K_t} = \frac{\frac{2}{3}K_t}{\frac{2}{3}K_t + K_t} = \frac{\frac{2}{3}}{\frac{5}{3}} = \frac{2}{5}$$

When rolling without slipping like this, the force of friction would be a static force of friction, and so it does not dissipate any energy. However, if an object is rolling while slipping, then the kinetic friction force between the object and the surface would cause energy to dissipate.

Example 12 A ring with a rotational inertia of MR^2 is sliding over a horizontal frictionless surface without rotating. Initially it has a translational speed, v_0 . It then moves onto a horizontal surface with a friction coefficient of μ_k . The kinetic friction force causes it to start rotating, and after rolling a distance L , the hoop starts to roll without slipping. What is the linear velocity of the hoop after traveling the distance L ?

Solution. Since the normal force must cancel the force of gravity on the ring, the force of friction can be found using $F_{f,k} = \mu_k F_N = \mu_k Mg$. From this we can find the work done by friction, which would be the amount of energy lost. Additionally, at the end it is rolling without slipping and we can once again use $\omega = v/r$.

$$E_i + W_{\text{other}} = E_f$$

$$\frac{1}{2}Mv_0^2 + (-\mu_k MgL) = \frac{1}{2}Mv_f^2 + \frac{1}{2}I\omega_f^2 = \frac{1}{2}Mv_f^2 + \frac{1}{2}MR^2\omega_f^2 = \frac{1}{2}Mv_f^2 + \frac{1}{2}Mv_f^2$$

$$\frac{1}{2}Mv_0^2 - \mu_k MgL = Mv_f^2$$

Therefore, the final velocity is

$$v_f = \sqrt{\frac{1}{2}v_0^2 - \mu_k gL}$$

CHAPTER 10 KEY TERMS

angular displacement
translational displacement
angular velocity
Right-Hand Rule
angular acceleration
center of mass
torque
net torque
translational equilibrium
rotational equilibrium
static equilibrium
rotational inertia
Parallel Axis Theorem
angular momentum
rotational kinetic energy

Chapter 10 Review Questions

Answers and explanations can be found in Chapter 12.

Section I: Multiple Choice

1  Mark for Review

A disc has a radius of 5 cm. If the disc rotates about its central axis at an angular speed of 8 rev/s, what is the linear speed of a point on the rim of the disc?

(A) 0.4 m/s

(B) 2.5 m/s

(C) 3.9 m/s

(D) 50 m/s

2  Mark for Review

An object, initially at rest, begins spinning under a constant angular acceleration. In 15 s, it completes an angular displacement of 90 rad. What is the magnitude of the angular acceleration it experiences?

(A) 0.4 rad/s²

(B) 0.8 rad/s²

(C) 6 rad/s²

(D) 12 rad/s²

3  Mark for Review

A wheel initially rotating at 12 rad/s decelerates uniformly to rest in 0.4 s. If the wheel has a rotational inertia of 0.5 kg·m², what is the magnitude of the torque causing this deceleration?

(A) 1.5 N·m

(B) 15 N·m

(C) 30 N·m

(D) 38 N·m

4  Mark for Review

A grinding wheel with a radius of 12 cm is being used to sharpen a knife. The side of the knife is pressed perpendicularly into the wheel's surface with a force of 16 N. The coefficient of kinetic friction between the knife and the wheel is 0.28. What is the torque the grinding wheel's motor must supply to maintain its rotation rate of 50 rad/s?

(A) 0.54 N·m

(B) 0.72 N·m

(C) 4.5 N·m

(D) 54 N·m

5  Mark for Review

A uniform 10-m ladder with a mass of 20 kg rests against a wall such that it makes a 50° angle with the ground. If the friction between the ladder and the wall is negligible, what is the force the ladder exerts on the wall?

(A) 82 N

(B) 98 N

(C) 164 N

(D) 196 N

6  Mark for Review

An object spins with an angular velocity ω . If the object's rotational inertia increases by a factor of 4, without the application of external torque, what will be the object's new angular velocity?

(A) $\omega/4$ (B) $\omega/2$ (C) 2ω (D) 4ω 7  Mark for Review

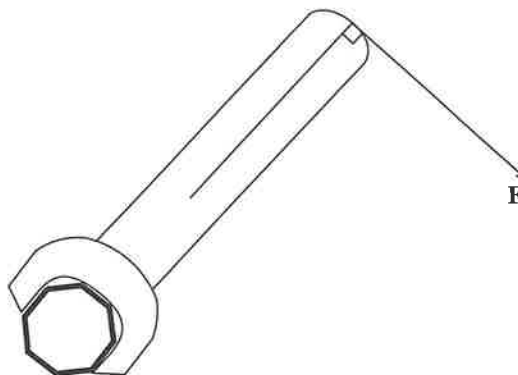
A child with a mass of 25 kg is standing at the edge of a 2.0 m radius merry-go-round, and they are rotating without friction at 20 rpm. The child then moves toward the center of the merry-go-round. The merry-go-round has a rotational inertia of $1200 \text{ kg}\cdot\text{m}^2$, and the rotational inertia of the child is $I = mr^2$, where r is her distance from the axis of rotation. What is the new angular velocity when she is 0.5 m from the center of the merry-go-round?

(A) 2.03 rad/s

(B) 2.09 rad/s

(C) 2.15 rad/s

(D) 2.25 rad/s

8  Mark for Review

In an effort to tighten a bolt, a force F is applied as shown in the figure above. If the distance from the end of the wrench to the center of the bolt is 20 cm and $F = 20 \text{ N}$, what is the magnitude of the torque produced by F ?

(A) 1 N·m

(B) 2 N·m

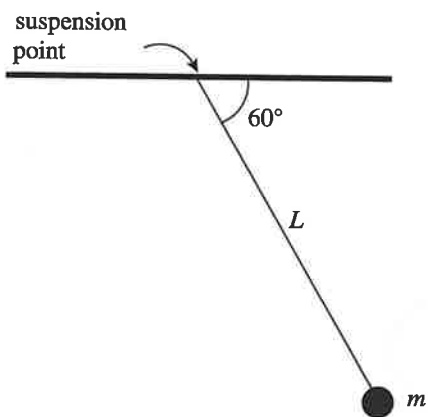
(C) 4 N·m

(D) 10 N·m

9



Mark for Review



In the figure above, what is the torque about the pendulum's suspension point produced by the weight of the bob, given that the length of the pendulum, L , is 80 cm and $m = 0.50$ kg?


(A) 0.5 N·m

(B) 1.0 N·m

(C) 1.7 N·m

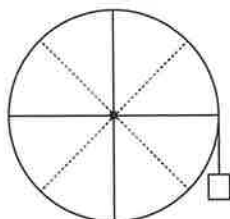
(D) 2.0 N·m

Section II: Free Response

1  Mark for Review

A student is tasked with determining the rotational inertia of a bicycle wheel.

Figure 1



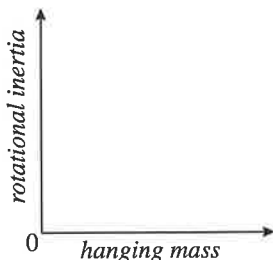
- A. The student plans to mount the wheel on a horizontal axis so that it can spin freely, and then hang a mass from a string wrapped around the outer diameter of the wheel.
- i. Design an experimental procedure to measure the rotational inertia. Assume equipment usually found in a school physics laboratory is available. In Table 1, list the quantities and associated symbols that would be measured in your experiment. Also, list the equipment that would be used to measure each quantity. You do not need to fill in every row. If you need additional rows, you may add them to the space just below the table.

Table 1

Quantity to be Measured	Symbol for Quantity	Equipment for Measurement

- ii. Determine an expression for the rotational inertia of the wheel, referring to the quantities listed in the table.
- B. Describe the overall procedure to be used to measure the rotational inertia of the wheel, referring to the table. Provide enough detail so that another student could replicate the experiment, including any steps necessary to reduce experimental uncertainty. As needed, use the symbols defined in the table and/or include a simple diagram of the setup.
- C. The student then decides to make a number of measurements by hanging different masses from the same wheel. On the axes in Figure 2, sketch a graph of the rotational inertia as a function of the hanging mass.

Figure 2



2  Mark for Review

A woman is riding her bicycle, which has wheels with a diameter D and a rotational inertia I .

- Assuming that there is negligible friction opposing the rotation of her wheels, derive an expression for the torque, τ_{app} , she applies to one of the wheels to accelerate from rest to a speed of v in a time t in terms of D , I , v , t , and physical constants, as appropriate.
- Suppose there is a kinetic friction force f acting on the wheel at a distance s from the wheel's rotation axis. If the torque the cyclist applies to the wheel remains the same, derive an expression for the time, t' , it will take her to attain a speed of v in terms of D , I , v , t , f , s , and physical constants, as appropriate.


3  Mark for Review

Figure 1



Two balls, A and B, of identical mass and radius are placed next to each other at the top of an incline and released from rest. The balls roll without slipping down the incline, along a horizontal surface, and then up another incline. Ball A reaches the bottom of the first incline with a greater translational speed than does Ball B. The balls end up reaching the same height on the second incline.

In terms of energy, explain the different translational speeds at the bottom of the first incline and the same final height of the two balls. Provide your answer in a clear, coherent paragraph-length response that may also contain figures and/or equations.

Chapter 10 Summary

- Torque is a property of a force that makes an object rotate. The equation for torque is $\tau = rF \sin \theta$. Torques may be clockwise or counterclockwise.
- The rotation of objects is described with rotational analogues to linear kinematic quantities: angular displacement $\Delta\theta$, angular velocity $\vec{\omega} = \frac{\Delta\theta}{\Delta t}$, and angular acceleration $\vec{\alpha} = \frac{\Delta\omega}{\Delta t}$.
- The Big 5 also applies to rotational motion, with the rotational quantities replacing the corresponding linear quantities.
- Torques result from forces applied at a distance from a pivot point, $\tau = rF \sin \theta$. Torques can be clockwise or counterclockwise and cause angular acceleration depending on the rotational inertia of the object, $\tau = I\alpha$.
- Rotational inertia reflects the distribution of an object's mass relative to an axis of rotation.
- An object is in equilibrium (not accelerating linearly or rotationally) when $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma \tau = 0$.
- The angular momentum of a system, $L = I\omega$, is conserved in the absence of outside net torque.
- A net torque applied to a rotating system over some time will change the angular momentum of the system, $\Delta L = \tau \Delta t$.
- A rotating object possesses rotational kinetic energy, $K_r = \frac{1}{2} I \omega^2$.

