Chapter 6 Circular Motion and Gravitations

"I can calculate the motion of heavenly bodies but not the madness of people."

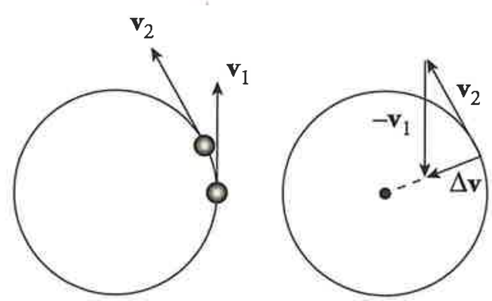
—Sir Isaac Newton

Since man has looked up to the sky, we have always tried to find a reason as to why celestial objects move the way they do. Kepler discovered that planets make elliptical orbits. Later, Newton realized the objects not moving in straight lines must experience an outside force. What force was making these orbits elliptical? Our first concepts explained linear motion. Then we added a second dimension and our motion became parabolic. Now we will explore when objects begin to undergo circular motion. Then we will have a better understanding of the orbit of our Moon around the Earth and the Earth around the Sun.

UNIFORM CIRCULAR MOTION

Let's simplify matters and consider the object's speed around its path to be constant. This is called **uniform circular motion**. You should remember that although the speed may be constant, the velocity is not because the direction of the velocity is always changing. Since the velocity is changing, there must be acceleration. This acceleration does not change the speed of the object; it changes only the direction of the velocity to keep the object on its circular path. Also, in order to produce an acceleration, there must be a force; otherwise, the object would move off in a straight line (Newton's First Law).

Take a look at the figures below. The figure on the left shows an object moving along a circular trajectory, along with its velocity vectors at two nearby points. The vector $\mathbf{v}_1$ is the object's velocity at time $t = t_1$, and $\mathbf{v}_2$ is the object's velocity vector a short time later (at time $t = t_2$). The velocity vector is always tangential to the object's path (whatever the shape of the trajectory). Notice that since we are assuming constant speed, the lengths of $\mathbf{v}_1$ and $\mathbf{v}_2$ (their magnitudes) are the same.



Non-Uniform Circular Objects

Planets and most objects do not undergo uniform circular motion. They usually follow elliptical orbits with varying speeds. For the purposes of the AP Physics 1 Exam, this will not be tested.

Since $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ points toward the center of the circle (see the figure on the right), so does the acceleration, since $\mathbf{a} = \Delta\mathbf{v}/\Delta t$. Because the acceleration vector points toward the center of the circle, it's called **centripetal acceleration**, or $\mathbf{a}_c$. The centripetal acceleration is what turns the velocity vector to keep the object traveling in a circle. The magnitude of the centripetal acceleration depends on the object's speed, v, and the radius, r, of the circular path according to the equation:

$$a_c = \frac{v^2}{r}$$

Example 1 An object of mass 5 kg moves at a constant speed of 6 m/s in a circular path of radius 2 m. Find the object's acceleration and the net force responsible for its motion.

Solution. By definition, an object moving at constant speed in a circular path is undergoing uniform circular motion. Therefore, it experiences a centripetal acceleration of magnitude v^2/r, always directed toward the center of the circle:

$$a_c = \frac{v^2}{r} = \frac{(6 \text{ m/s})^2}{2 \text{ m}} = 18 \text{ m/s}^2$$

The force that produces the centripetal acceleration is given by Newton's Second Law, coupled with the equation for centripetal acceleration:

$$F_c = ma_c = m\frac{v^2}{r}$$

This equation gives the magnitude of the force. As for the direction, recall that because **F** = m**a**, the directions of **F** and **a** are always the same. Since centripetal acceleration points toward the center of the circular path, so does the force that produces it. Therefore, it's called **centripetal force.** The centripetal force acting on this object has a magnitude of $F_c = ma_c = (5 \text{ kg})(18 \text{ m/s}^2) = 90 \text{ N}$.

Example 2 A 10.0 kg mass is attached to a string that has a breaking strength of 200 N. If the mass is whirled on a frictionless surface in a horizontal circle of radius 80 cm, what maximum speed can it have?

Solution. The first thing to do in problems like this is to identify what forces produce the centripetal acceleration. Notice that this is a horizontal circle. We can limit our examination to the horizontal (x) direction. Because gravity and the normal force exert forces in the y direction, they can be ignored. If we were given a problem with a vertical circle, we would have to include the effects of gravity, which will be demonstrated in Example 4. In this example, the tension in the string produces the centripetal force:

$$\mathbf{F}_\text{T} \text{ provides } \mathbf{F}_c \Rightarrow F_\text{T} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{rF_\text{T}}{m}} \Rightarrow v_{max} = \sqrt{\frac{rF_{\text{T, max}}}{m}}$$

$$= \sqrt{\frac{(0.80 \text{ m})(200 \text{ N})}{10 \text{ kg}}}$$

$$= 4 \text{ m/s}$$

Notice the change in units from 80 cm to 0.80 m. As a general rule, stick to kilograms, meters, and seconds because the newton is composed of these units.

Example 3 An athlete who weighs 800 N is running around a curve at a speed of 5.0 m/s in an arc whose radius of curvature, r, is 5.0 m. Find the centripetal force acting on him. What provides the centripetal force? What could happen to him if r were smaller?

Such a Slacker
Centripetal force is the only force you need to know for the AP Physics 1 Exam that can NEVER do work. This is because, by definition, it is always perpendicular to the direction of the object's motion.

Centripetal Force and Centrifugal Force
Centripetal force points into the center of the circle and centrifugal points away from the circle. Centrifugal force is referred to as a fictitious force since it is not a real force. Centripetal force is the net force from the physical forces acting on the object. Neither one should ever be drawn on a force diagram.

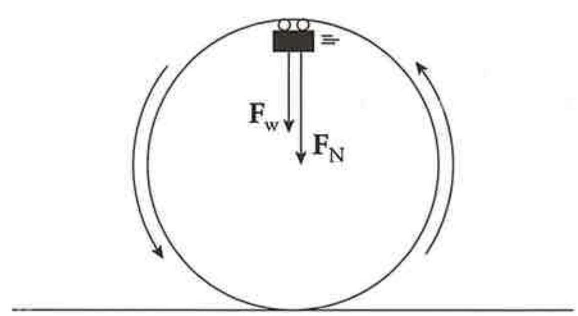
Solution. Using the equation for the strength of the centripetal force, we find that

$$F_c = m\frac{v^2}{r} = \frac{F_w}{g}\cdot\frac{v^2}{r} = \frac{800\ \text{N}}{10\ \text{N/kg}}\cdot\frac{(5.0\ \text{m/s})^2}{5.0\ \text{m}} = 400\ \text{N}$$

In this case, static friction provides the centripetal force. If the radius of curvature of the arc were smaller, then the centripetal force required to keep him running in a circle would increase. If the centripetal force increased enough, it might exceed what the force of static friction could provide, at which point he would slip.

Example 4 A roller-coaster car enters the circular-loop portion of the ride. At the very top of the circle (where the people in the car are upside down), the speed of the car is 15 m/s, and the acceleration points straight down. If the diameter of the loop is 40 m and the total mass of the car (plus passengers) is 1200 kg, find the magnitude of the normal force exerted by the track on the car at this point.

Solution. There are two forces acting on the car at its topmost point: the normal force exerted by the track and the gravitational force, both of which point downward. At the top of the loop, the gravitational force and the normal force point downward. This is because the normal force acts perpendicular to the surface of the track, and the gravitational force is always directed downward.



The combination of these two forces, $\mathbf{F}_N + \mathbf{F}_w$, provides the centripetal force:

$$F_N + F_w = \frac{mv^2}{r} \Rightarrow F_N = \frac{mv^2}{r} - F_w$$

$$= \frac{mv^2}{r} - mg$$

$$= m\left(\frac{v^2}{r} - g\right)$$

$$= (1200\ \text{kg})\left[\frac{(15\ \text{m/s})^2}{\frac{1}{2}(40\ \text{m})} - 10\ \text{m/s}^2\right]$$

$$= 1500\ \text{N}$$

Example 5 In the previous example, if the net force on the car at its topmost point is straight down, why doesn't the car fall straight down?

Solution. Remember that force tells an object how to accelerate. If the car had zero velocity at this point, then it would certainly fall straight down, but the car has a nonzero velocity (to the left) at this point. The fact that the acceleration is downward means that, at the next moment, v will point down to the left at a slight angle, ensuring that the car remains on a circular path, in contact with the track.

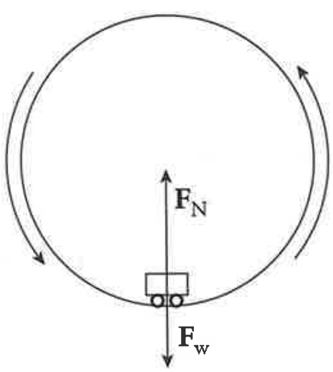
Example 6 How would the normal force change in Example 4 if the car was at the bottom of the circle?

Solution. There are still two forces acting on the car: the gravitational force still points downward, but the normal force pushes 90 degrees to the surface (upward). These forces now oppose one another. The combination of these two forces still provides the centripetal force. Because the centripetal acceleration points inward, we will make anything that points toward the center of the circle positive and anything that points away from the circle negative. Therefore, our equation becomes

$$F_N - F_w = \frac{mv^2}{r} \Rightarrow F_N = \frac{mv^2}{r} + F_w$$

$$= \frac{mv^2}{r} + mg$$

$$= m\left(\frac{v^2}{r} + g\right)$$



$$= 1200 \text{ kg}\left[\frac{(15 \text{ m/s})^2}{\frac{1}{2}(40 \text{ m})} + 10 \text{ m/s}^2\right]$$

$$= 25{,}500 \text{ N}$$

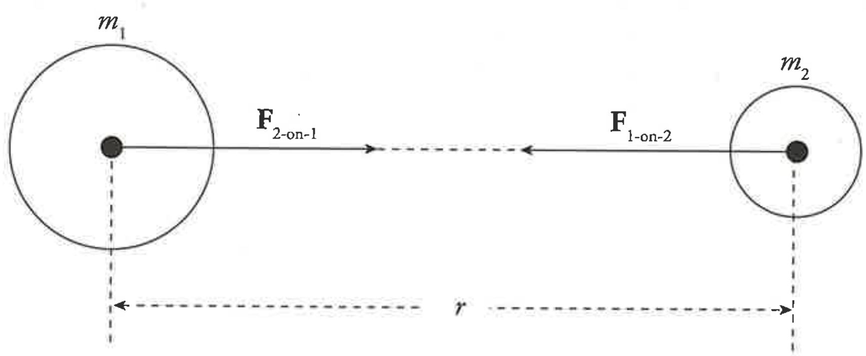
Notice the big difference between this answer and the answer from Example 4. This is why you would feel very little force between you and the seat at the top of the loop, but you would feel a big slam at the bottom of the loop.

NEWTON'S LAW OF GRAVITATION

A Mass by Any Other Name
Technically, the masses in Newton's Law of Gravitation refer to gravitational mass, whereas the one in Newton's Second Law is an inertial mass. Fortunately, they've been experimentally verified to be the same, so we don't have to keep track of the difference!

Newton eventually formulated a law of gravitation: any two objects in the universe exert an attractive force on each other—called the **gravitational force**—whose strength is proportional to the product of the objects' masses and inversely proportional to the square of the distance between them as measured from center to center. If we let G be the **universal gravitational constant**, then the strength of the gravitational force is given by the following equation:

$$F_G = \frac{G m_1 m_2}{r^2}$$



Gravity is always a pulling force.

The forces $\mathbf{F}_{1\text{-on-}2}$ and $\mathbf{F}_{2\text{-on-}1}$ act along the line that joins the bodies and form an action-reaction pair.

The first reasonably accurate numerical value for G was determined by Cavendish more than 100 years after Newton's law was published. To two decimal places, the currently accepted value of G is

$$G = 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$$

Close Enough
Some might wonder why 10 m/s² is used for gravity everywhere on Earth. What if we're on top of a mountain? The difference between sea level and the peak of Mount Everest is less than half a percent. Super-accurate measurement nitpicking is for chemistry class, not physics.

Why is g equal to 9.8 m/s²? Generally, the acceleration due to gravity on an object of m at a distance r from a planet of mass M is

$$\vec{g} = \frac{\vec{F}_g}{m} = \frac{G\frac{Mm}{r^2}}{m} = \frac{GM}{r^2}$$

The force of gravity is attractive, so the direction of acceleration is toward the center of the planet. The acceleration due to gravity $\vec{g}$ is also known as a gravitational field. At any point in space, multiplying $\vec{g}$ by the mass of an object at that point gives the gravitational force acting on that object. Since the Earth is a sphere, r is approximately equal to the radius of the Earth at all points on its surface. Using the mass of the Earth and the radius of the Earth gives a magnitude of 9.8 m/s^2.

Example 7 Given that the radius of the Earth is 6.37×10^6 m, determine the mass of the Earth.

Solution. Consider a small object of mass m near the surface of the Earth (mass M). Its weight is mg, but its weight is just the gravitational force it feels due to the Earth, which is GMm/R^2. Therefore,

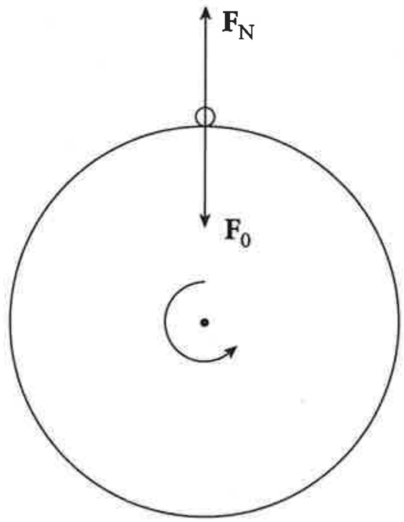
$$mg = G\frac{Mm}{R^2} \quad \Rightarrow \quad M = \frac{gR^2}{G}$$

Since we know that $g = 10$ m/s^2 and $G = 6.67 \times 10^{-11}$ N·m^2/kg^2, we can substitute to find

$$M = \frac{gR^2}{G} = \frac{(10 \text{ m/s}^2)(6.37 \times 10^6 \text{ m})^2}{6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2} = 6.1 \times 10^{24} \text{ kg}$$

Example 8 We can derive the expression $g = GM/R^2$ by equating mg and GMm/R^2 (as we did in the previous example), and this gives the magnitude of the *absolute gravitational acceleration*, a quantity that's sometimes denoted as g_0. The notation g is acceleration, but with the spinning of the Earth taken into account. Show that if an object is at the equator, its *measured weight* (that is, the weight that a scale would measure), mg, is less than its *true weight*, mg_0, and compute the weight difference for a person of mass $m = 60$ kg.

Solution. Imagine looking down at the Earth from above the North Pole.



Relationships Are Important
Not just in daily life but especially in physics. Recognizing the relationship between variables in formulas means, at times, you will not need to do much calculation.

The net force toward the center of the Earth is $\mathbf{F}_0 - \mathbf{F}_N$, which provides the centripetal force on the object. Therefore,

$$F_0 - F_N = \frac{mv^2}{R}$$

Since $v = \dfrac{2\pi R}{T}$, where T is the Earth's rotation period, we have

$$F_0 - F_N = \frac{m}{R}\left(\frac{2\pi R}{T}\right)^2 = \frac{4\pi^2 mR}{T^2}$$

or, since $F_0 = mg_0$ and $F_N = mg$,

$$mg_0 - mg = \frac{4\pi^2 mR}{T^2}$$

Since the quantity $\dfrac{4\pi^2 mR}{T^2}$ is positive, mg must be less than mg_0. The difference between mg_0 and mg, for a person of mass $m = 60$ kg, is only

$$\frac{4\pi^2 mR}{T^2} = \frac{4\pi^2(60\text{ kg})(6.37\times10^6\text{ m})}{\left(24\text{ hr}\times\frac{60\text{ min}}{\text{hr}}\times\frac{60\text{ s}}{\text{min}}\right)^2} = 2.0\text{ N}$$

and the difference between g_0 and g is

$$g_0 - g = \frac{mg_0 - mg}{m} = \frac{4\pi^2 R}{T^2} = \frac{4\pi^2(6.37\times10^6\text{ m})}{\left(24\text{ hr}\times\frac{60\text{ min}}{\text{hr}}\times\frac{60\text{ s}}{\text{min}}\right)^2} = 0.034\text{ m/s}^2$$

Note that this difference is so small (< 0.3%) that it can usually be ignored.

Example 9 Communications satellites are often parked in geosynchronous orbits above Earth's surface. These satellites have orbit periods that are equal to Earth's rotation period, so they remain above the same position on Earth's surface. Determine the altitude that a satellite must have to be in a geosynchronous orbit above a fixed point on Earth's equator. (The mass of the Earth is 5.98×10^{24} kg.)

Solution. Let m be the mass of the satellite, M the mass of Earth, and R the distance from the center of Earth to the position of the satellite. The gravitational pull of Earth provides the centripetal force on the satellite, so:

$$G\frac{Mm}{R^2} = \frac{mv^2}{R} \quad \Rightarrow \quad G\frac{M}{R} = v^2$$

The orbit speed of the satellite is $2\pi R/T$, so:

$$G\frac{M}{R} = \left(\frac{2\pi R}{T}\right)^2$$

which implies:

$$G\frac{M}{R} = \frac{4\pi^2 R^2}{T^2} \quad \Rightarrow \quad 4\pi^2 R^3 = GMT^2 \quad \Rightarrow \quad R = \sqrt[3]{\frac{GMT^2}{4\pi^2}}$$

Now the key feature of a geosynchronous orbit is that its period matches Earth's rotation period, $T = 24$ hr. Substituting the numerical values of G, M, and T into this expression, we find that

$$R = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = \sqrt[3]{\frac{(6.67\times10^{-11})(5.98\times10^{24})(24\cdot60\cdot60)^2}{4\pi^2}}$$
$$= 4.23\times10^7 \text{ m}$$

Therefore, if r_E is the radius of Earth, then the satellite's altitude above Earth's surface must be

$$h = R - r_E = (4.23 \times 10^7 \text{ m}) - (6.37 \times 10^6 \text{ m}) = 3.59 \times 10^7 \text{ m}$$

Example 10 The Moon orbits Earth in a (nearly) circular path at (nearly) constant speed. If M is the mass of Earth, m is the mass of the Moon, and r is the Moon's orbit radius, find an expression for the Moon's orbit speed.

Solution: We begin by answering the question, "What produces the centripetal force?" The answer is the gravitational pull by the Earth. We now simply translate our answer into an equation, like this:

gravitational pull	produces	the centripetal force
F_{grav}	$=$	$m\left(\frac{v^2}{r}\right)$

Since we know $F_{grav} = \left(\frac{GMm}{r^2}\right)$, we get

$$F_{grav} = F_c$$

$$G\left(\frac{Mm}{r^2}\right) = m\left(\frac{v^2}{r}\right)$$

$$G\left(\frac{M}{r}\right) = v^2$$

$$v = \left(\frac{GM}{r}\right)^{\frac{1}{2}}$$

Notice that the mass of the Moon, m, cancels out. So, any object orbiting at the same distance from the Earth as the Moon must move at the same speed as the Moon.

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CHAPTER 6 KEY TERMS

uniform circular motion
centripetal acceleration
centripetal force
Newton's Law of Gravitation
gravitational force
universal gravitational constant

Chapter 6 Review Questions

Answers and explanations can be found in Chapter 12.

Section I: Multiple Choice

1 Mark for Review

An object moves at constant speed in a circular path. Which of the following statements is true?

(A) The velocity is changing.

(B) The velocity is constant.

(C) The acceleration is non-zero and constant.

(D) The acceleration is zero.

Questions 2 through 3 refer to the following.

A 60 cm rope is tied to the handle of a bucket which is then whirled in a vertical circle. The mass of the bucket is 3 kg.

2 Mark for Review

At the lowest point in its path, the tension in the rope is 50 N. What is the speed of the bucket?

(A) 1 m/s

(B) 2 m/s

(C) 3 m/s

(D) 4 m/s

3 Mark for Review

What is the critical speed below which the rope would become slack when the bucket reaches the highest point in the circle?

(A) 0.6 m/s

(B) 1.8 m/s

(C) 2.4 m/s

(D) 4.8 m/s

4 Mark for Review

An object moves at a constant speed in a circular path of radius r at a rate of 1 revolution per second. What is its acceleration?

(A) 0

(B) $2\pi^2 r$

(C) $2\pi^2 r^2$

(D) $4\pi^2 r$

5 Mark for Review

If the distance between two point particles is doubled, then the gravitational force between them

(A) decreases by a factor of 4

(B) decreases by a factor of 2

(C) increases by a factor of 2

(D) increases by a factor of 4

6 Mark for Review

At the surface of Earth, an object of mass m has weight w. If this object is transported to an altitude that is twice the radius of Earth, then at the new location,

(A) its mass is m and its weight is $w/2$

(B) its mass is $m/2$ and its weight is $w/4$

(C) its mass is m and its weight is $w/4$

(D) its mass is m and its weight is $w/9$

7 Mark for Review

A moon of mass m orbits a planet of mass $100m$. Let the strength of the gravitational force exerted by the planet on the moon be denoted by F_1, and let the strength of the gravitational force exerted by the moon on the planet be F_2. Which of the following is true?

(A) $F_1 = 100F_2$

(B) $F_1 = 10F_2$

(C) $F_1 = F_2$

(D) $F_2 = 10F_1$

8 Mark for Review

The dwarf planet Pluto has 1/500 the mass and 1/15 the radius of Earth. What is the value of g (in m/s^2) on the surface of Pluto?

(A) $\frac{50}{225}$

(B) $\frac{50}{15}$

(C) $\frac{15}{50}$

(D) $\frac{225}{50}$

9 Mark for Review

A moon of Jupiter has a nearly circular orbit of radius R and an orbit period of T. Which of the following expressions gives the mass of Jupiter?

(A) $\frac{4\pi^2 R}{T^2}$

(B) $\frac{2\pi R^3}{(GT^2)}$

(C) $\frac{4\pi R^2}{(GT^2)}$

(D) $\frac{4\pi^2 R^3}{(GT^2)}$

10 Mark for Review

Two large bodies, Body A of mass m and Body B of mass $4m$, are separated by a distance R. At what distance from Body A, along the line joining the bodies, would the gravitational force on an object be equal to zero? (Ignore the presence of any other bodies.)

(A) $\frac{R}{16}$

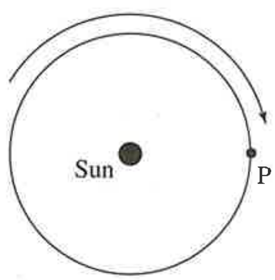
(B) $\frac{R}{8}$

(C) $\frac{R}{4}$

(D) $\frac{R}{3}$

11 Mark for Review

You are looking at a top view of a planet orbiting the Sun in a clockwise direction. Which of the following would describe the velocity, acceleration, and force acting on the planet due to the Sun's pull at point P?



(A) $v\downarrow$ $a\uparrow$ $F\uparrow$

(B) $v\downarrow$ $a\leftarrow$ $F\leftarrow$

(C) $v\downarrow$ $a\rightarrow$ $F\rightarrow$

(D) $v\uparrow$ $a\leftarrow$ $F\leftarrow$

12 Mark for Review

Which of the following statements is true for a satellite in outer space orbiting the Earth in uniform circular motion?

(A) There are no forces acting on the satellite.

(B) The force of gravity is the only force acting on the satellite.

(C) The force of gravity is balanced by the outward force of the object.

(D) The outward force of the object is the only force acting on the satellite.

Section II: Free Response

1 Mark for Review

A robot probe lands on a new, uncharted planet. It has determined the diameter of the planet to be 8×10^6 m. It weighs a standard 1 kg mass and determines that 1 kg weighs only 5 newtons on this new planet.

A. What must the mass of the planet be?

B. What is the acceleration due to gravity on this planet? Express your answer in both m/s^2 and g's (where 1 g = 10 m/s^2).

C. What is the average density of this planet?

2 Mark for Review

The Earth has a mass of 6×10^{24} kg and orbits the Sun in 3.15×10^7 seconds at a constant radius of 1.5×10^{11} m.

A. What is the Earth's centripetal acceleration around the Sun?

B. What is the gravitational force acting between the Sun and Earth?

C. What is the mass of the Sun?

3 Mark for Review

An amusement park ride consists of a large cylinder that rotates around its central axis as the passengers stand against the inner wall of the cylinder. Once the passengers are moving at a certain speed, v, the floor on which they are standing is lowered. Each passenger feels pinned against the wall of the cylinder as it rotates. Let r be the inner radius of the cylinder.

A. Draw and label all the forces acting on a passenger of mass m as the cylinder rotates with the floor lowered.

B. Describe what conditions must hold to keep the passengers from sliding down the wall of the cylinder.

C. Compare the conditions discussed in part B for an adult passenger of mass m and a child passenger of mass $m/2$.

4 Mark for Review

A curved section of a highway has a radius of curvature of r. The coefficient of friction between standard automobile tires and the surface of the highway is μ_s.

A. Draw and label all the forces acting on a car of mass m traveling along this curved part of the highway.

B. Compute the maximum speed with which a car of mass m could make it around the turn without skidding in terms of μ_s, r, g, and m.

City engineers are planning to bank this curved section of highway at an angle of θ to the horizontal.

C. Draw and label all of the forces acting on a car of mass m traveling along this banked turn. Do not include friction.

D. The engineers want to be sure that a car of mass m traveling at a constant speed v (the posted speed limit) could make it safely around the banked turn even if the road were covered with ice (that is, essentially frictionless). Compute this banking angle θ in terms of r, v, g, and m.

Chapter 6 Summary

- For objects undergoing uniform circular motion, the centripetal acceleration is given by $a_c = \frac{v^2}{r}$ and the centripetal force is given by $F_c = \frac{mv^2}{r}$.
- For any two masses in the universe, there is a gravitational attraction given by

$$F_G = \frac{Gm_1m_2}{r^2} \text{ where}$$

$$G = 6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}$$

- The acceleration due to gravity on any planet is given by

$$g_{planet} = \frac{Gm_{planet}}{r^2}$$

- Many times, universal gravitation is linked up with circular motion (because planetary orbits are very nearly circular). Therefore, it is useful to keep the following equations for circular motion mentally linked for those questions that include orbits:

$$v = \frac{2\pi r}{T} \qquad a_c = \frac{v^2}{r} \qquad a_c = \frac{4\pi^2 r}{T^2}$$

$$F = ma_c \qquad F_c = \frac{mv^2}{r} \qquad F_c = \frac{4\pi^2 mr}{T^2}$$