AMC 10

SET 1

PRACTICE EXAMINATION

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 25 problems are taken from past AMC 10 contests (2000–2019) and appear in their original wording, rearranged into a new paper. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**C**	2006 AMC 10B #1	Algebra
2	**C**	2016 AMC 10A #2	Algebra
3	**B**	2003 AMC 10B #3	Algebra
4	**B**	2015 AMC 10A #4	Algebra
5	**D**	2019 AMC 10A #5	Number Theory
6	**D**	2018 AMC 10B #6	Counting & Probability
7	**B**	2005 AMC 10B #7	Geometry
8	**B**	2017 AMC 10A #8	Counting & Probability
9	**C**	2014 AMC 10A #9	Geometry
10	**C**	2002 AMC 10B #10	Algebra
11	**C**	2007 AMC 10A #11	Counting & Probability
12	**A**	2004 AMC 10B #12	Geometry
13	**A**	2011 AMC 10A #13	Counting & Probability
14	**D**	2014 AMC 10B #14	Number Theory
15	**E**	2009 AMC 10A #15	Algebra
16	**A**	2017 AMC 10B #16	Number Theory
17	**A**	2010 AMC 10A #17	Geometry
18	**B**	2008 AMC 10B #18	Algebra
19	**C**	2013 AMC 10A #19	Number Theory
20	**A**	2012 AMC 10A #20	Counting & Probability
21	**A**	2008 AMC 10A #21	Geometry
22	**B**	2007 AMC 10B #22	Counting & Probability
23	**D**	2018 AMC 10A #23	Geometry
24	**B**	2011 AMC 10B #24	Geometry
25	**B**	2010 AMC 10B #25	Algebra

1 Algebra

Source: 2006 AMC 10B #1

What is $(-1)^1 + (-1)^2 + \cdots + (-1)^{2006}$?

(A) -2006 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2006

Key idea. The powers of -1 simply alternate: an odd exponent gives -1 and an even exponent gives $+1$. So pair the terms off two at a time.

The terms are $-1,\ +1,\ -1,\ +1,\ \ldots$, ending with $(-1)^{2006} = +1$. Group them in consecutive pairs:

$$(-1+1) + (-1+1) + \cdots + (-1+1).$$

Each pair adds to 0. There are 2006 terms, and 2006 is even, so they form exactly 1003 pairs with nothing left over. The total is $1003 \cdot 0 = 0$.

Answer: (C) 0

2 Algebra

Source: 2016 AMC 10A #2

For what value of x does

$$10^x \cdot 100^{2x} = 1000^5?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Key idea. Write every number as a power of the same base, 10; then the exponents must be equal.

Step 1. Since $100 = 10^2$, we have $100^{2x} = (10^2)^{2x} = 10^{4x}$. So the left side is

$$10^x \cdot 10^{4x} = 10^{x+4x} = 10^{5x}.$$

Step 2. Since $1000 = 10^3$, the right side is $1000^5 = (10^3)^5 = 10^{15}$.

Step 3. Now $10^{5x} = 10^{15}$, so $5x = 15$ and $x = 3$.

Check: $10^3 \cdot 100^6 = 10^3 \cdot 10^{12} = 10^{15} = 1000^5$.

Watch out. Forgetting to convert the base 100 and treating 100^{2x} as 10^{2x} gives $10^{3x} = 10^{15}$, so $x = 5$, choice (E). A power of a power multiplies the exponents: $(10^2)^{2x} = 10^{4x}$.

Answer: (C) 3

3 Algebra

Source: 2003 AMC 10B #3

> The sum of 5 consecutive even integers is 4 less than the sum of the first 8 consecutive odd counting numbers. What is the smallest of the even integers?
>
> **(A)** 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

Key idea. Find the target sum first, then write the five even integers in terms of the smallest one.

Step 1. The first 8 odd counting numbers are $1, 3, 5, \ldots, 15$, and

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 64.$$

(The sum of the first n odd numbers is always n^2, and $8^2 = 64$.)

Step 2. The five even integers sum to 4 less: $64 - 4 = 60$.

Step 3. Call the smallest one n. The five are n, $n+2$, $n+4$, $n+6$, $n+8$, with sum $5n + 20$. So $5n + 20 = 60$, giving $5n = 40$ and $n = 8$.

Check: $8 + 10 + 12 + 14 + 16 = 60$.

Watch out. Dividing 60 by 5 gives 12, which is the *average* of the five numbers — the middle one, not the smallest. Stopping there gives choice (D).

Answer: (B) 8

4 Algebra

Source: 2015 AMC 10A #4

> Pablo, Sofia, and Mia got some candy eggs at a party. Pablo had three times as many eggs as Sofia, and Sofia had twice as many eggs as Mia. Pablo decides to give some of his eggs to Sofia and Mia so that all three will have the same number of eggs. What fraction of his eggs should Pablo give to Sofia?
>
> **(A)** $\frac{1}{12}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

Key idea. Give Mia, who has the fewest, a variable m; everyone else is a multiple of m. Then find the fair share.

Step 1. Mia has m eggs, Sofia has twice that, $2m$, and Pablo has three times Sofia's, $6m$.

Step 2. The total is $m + 2m + 6m = 9m$, so for all three to be equal each must end with $9m \div 3 = 3m$.

Step 3. Sofia needs $3m - 2m = m$ more eggs, and these all come from Pablo. As a fraction of Pablo's eggs this is

$$\frac{m}{6m} = \frac{1}{6}.$$

(Mia gets $2m$ and Pablo keeps $6m - m - 2m = 3m$, so everything checks.)

Watch out. The question asks for a fraction of *Pablo's* eggs going to *Sofia*. Using Mia's share instead gives $\frac{2m}{6m} = \frac{1}{3}$, choice (D); using everything Pablo gives away gives $\frac{3m}{6m} = \frac{1}{2}$, choice (E).

Answer: (B) $\frac{1}{6}$

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5 Number Theory

Source: 2019 AMC 10A #5

What is the greatest number of consecutive integers whose sum is 45?

(A) 9 **(B)** 25 **(C)** 45 **(D)** 90 **(E)** 120

Key idea. "Integers" includes negatives. A long run of negatives can cancel a matching run of positives, so look for a run that is mostly cancellation.

Step 1 (a run that works). Take $-44, -43, \ldots, -1, 0, 1, \ldots, 44, 45$. Everything from -44 to 44 cancels in pairs (-44 with 44, and so on, plus 0), leaving 45. The run has $44 + 1 + 45 = 90$ terms.

Step 2 (nothing longer works). Suppose k consecutive integers start at a. Their sum is the number of terms times the average of the first and last:

$$\frac{k\,(2a + k - 1)}{2} = 45, \quad \text{so} \quad k\,(2a + k - 1) = 90.$$

Since $2a + k - 1$ is an integer, k must divide 90. Hence $k \leq 90$, and 120 is impossible.

Watch out. If you only think of positive integers, the longest run is $1 + 2 + \cdots + 9 = 45$, which has 9 terms — choice (A). Allowing negative numbers changes everything.

Answer: (D) 90

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6 Counting & Probability

Source: 2018 AMC 10B #6

A box contains 5 chips, numbered 1, 2, 3, 4, and 5. Chips are drawn randomly one at a time without replacement until the sum of the values drawn exceeds 4. What is the probability that 3 draws are required?

(A) $\frac{1}{15}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{5}$ **(E)** $\frac{1}{4}$

Key idea. Exactly 3 draws are needed precisely when the first two chips add up to 4 or less. So count the ordered pairs of first two chips.

Step 1. Which first two chips keep the sum at most 4? Only $\{1, 2\}$ (sum 3) and $\{1, 3\}$ (sum 4). Then neither the first chip alone nor the first two reach 5, while every chip left is at least 2, so the third draw pushes the sum to at least 6 — exactly 3 draws.

Step 2. Imagine the whole order of the five chips is shuffled in advance. The first two chips form one of $5 \cdot 4 = 20$ equally likely ordered pairs (even if the drawing would really stop after one chip). The good ordered pairs are

$$(1,2),\ (2,1),\ (1,3),\ (3,1),$$

which is 4 of them.

Step 3. The probability is $\dfrac{4}{20} = \dfrac{1}{5}$.

Watch out. Mixing ordered and unordered counts: taking the 2 unordered pairs $\{1,2\}, \{1,3\}$ over the 20 *ordered* pairs gives $\frac{2}{20} = \frac{1}{10}$, choice (B). Count both the same way — 2 out of $\binom{5}{2} = 10$ unordered pairs also gives $\frac{1}{5}$.

Answer: (D) $\dfrac{1}{5}$

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7 Geometry

Source: 2005 AMC 10B #7

A circle is inscribed in a square, then a square is inscribed in this circle, and finally, a circle is inscribed in this square. What is the ratio of the area of the smaller circle to the area of the larger square?

(A) $\dfrac{\pi}{16}$ **(B)** $\dfrac{\pi}{8}$ **(C)** $\dfrac{3\pi}{16}$ **(D)** $\dfrac{\pi}{4}$ **(E)** $\dfrac{\pi}{2}$

Key idea. Start from the innermost shape, give it a radius r, and work outward one shape at a time. The key link is that a square inscribed in a circle has the circle's diameter as its diagonal.

Step 1. Let the smaller circle have radius r; its area is πr^2. It is inscribed in the smaller square, so that square has side $2r$.

Step 2. The smaller square is inscribed in the larger circle, so its diagonal, $2r\sqrt{2}$, is the larger circle's diameter. The larger circle has radius $r\sqrt{2}$.

Step 3. The larger circle is inscribed in the larger square, so that square has side $2r\sqrt{2}$ and area $(2r\sqrt{2})^2 = 8r^2$.

Step 4. The ratio is $\dfrac{\pi r^2}{8r^2} = \dfrac{\pi}{8}$.

Watch out. Comparing the smaller circle with the *smaller* square gives $\frac{\pi r^2}{4r^2} = \frac{\pi}{4}$, choice (D). The question asks for the larger square.

Answer: (B) $\dfrac{\pi}{8}$

8 Counting & Probability

Source: 2017 AMC 10A #8

At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?

(A) 240 **(B)** 245 **(C)** 290 **(D)** 480 **(E)** 490

Key idea. Every pair of people either hugs or shakes hands. Only pairs that include at least one of the 10 strangers shake hands, so split those pairs into two kinds.

Step 1 (stranger with one of the 20). Each of the 10 strangers shakes hands with each of the 20: $10 \cdot 20 = 200$ handshakes.

Step 2 (two strangers). The 10 strangers know no one, so every pair of them shakes hands: $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ handshakes.

Step 3. Total: $200 + 45 = 245$.

Another way. Count all pairs and remove the hugs: $\binom{30}{2} - \binom{20}{2} = 435 - 190 = 245$.

Watch out. Counting "each stranger shakes 9 other strangers' hands" as $10 \cdot 9 = 90$ counts every stranger–stranger handshake twice, giving $200 + 90 = 290$, choice (C).

Answer: (B) 245

9 Geometry

Source: 2014 AMC 10A #9

The two legs of a right triangle, which are altitudes, have lengths $2\sqrt{3}$ and 6. How long is the third altitude of the triangle?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Key idea. The third altitude goes to the hypotenuse. Compute the area two ways: once with the legs as base and height, once with the hypotenuse as base.

Step 1. Using the legs, the area is $\frac{1}{2} \cdot 2\sqrt{3} \cdot 6 = 6\sqrt{3}$.

Step 2. By the Pythagorean Theorem the hypotenuse is

$$\sqrt{(2\sqrt{3})^2 + 6^2} = \sqrt{12 + 36} = \sqrt{48} = 4\sqrt{3}.$$

Step 3. If h is the altitude to the hypotenuse, then $\frac{1}{2} \cdot 4\sqrt{3} \cdot h = 6\sqrt{3}$, so $2\sqrt{3}\,h = 6\sqrt{3}$ and $h = 3$.

(Shortcut: the altitude to the hypotenuse equals $\dfrac{\text{leg}\cdot\text{leg}}{\text{hypotenuse}} = \dfrac{2\sqrt{3}\cdot 6}{4\sqrt{3}} = 3$.)

Answer: (C) 3

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10 Algebra

Source: 2002 AMC 10B #10

> Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. What is the pair (a, b)?
>
> **(A)** $(-2, 1)$ **(B)** $(-1, 2)$ **(C)** $(1, -2)$ **(D)** $(2, -1)$ **(E)** $(4, 4)$

Key idea. When the roots of a quadratic are named, Vieta's formulas turn them into equations: for $x^2 + px + q = 0$, the roots add to $-p$ and multiply to q.

Step 1. The roots of $x^2 + ax + b = 0$ are a and b, so

$$a + b = -a \qquad \text{and} \qquad ab = b.$$

Step 2. From $ab = b$ we get $b(a - 1) = 0$. Since $b \neq 0$, $a = 1$.

Step 3. Then $a + b = -a$ becomes $1 + b = -1$, so $b = -2$.

Check: $x^2 + x - 2 = (x - 1)(x + 2)$, whose roots are 1 and -2 — exactly a and b.

Another way. Test the choices. Each pair must make both a and b roots. For $(1, -2)$: $1^2 + 1 \cdot 1 - 2 = 0$ and $(-2)^2 + 1 \cdot (-2) - 2 = 0$. For $(-2, 1)$, plugging in $x = -2$ gives $4 + 4 + 1 = 9 \neq 0$; the other choices fail in the same way.

Answer: (C) $(1, -2)$

11 Counting & Probability

Source: 2007 AMC 10A #11

The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 24

Key idea. We are not told *how* the numbers are placed, only that all face sums are equal. That is a signal to add up all the face sums at once and count how often each number gets used.

Step 1. Call the common face sum S. Adding the sums of all six faces gives $6S$.

Step 2. Now count the same total vertex by vertex. Every vertex of a cube lies on exactly 3 faces, so each of the numbers $1, 2, \ldots, 8$ is counted 3 times:

$$6S = 3(1 + 2 + \cdots + 8) = 3 \cdot 36 = 108.$$

Step 3. So $S = 108 \div 6 = 18$.

Such placements really exist — for example, put $1, 4, 6, 7$ on the bottom face (sum 18) and $8, 5, 3, 2$ on the top face, each directly above $1, 4, 6, 7$ respectively; every side face then pairs a bottom edge with the top edge above it, and $1 + 4 + 8 + 5$, $4 + 6 + 5 + 3$, $6 + 7 + 3 + 2$, $7 + 1 + 2 + 8$ are all 18.

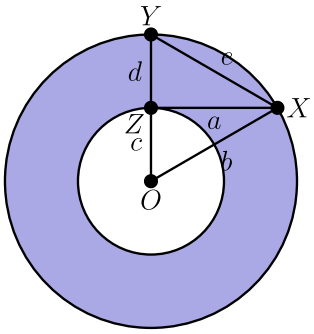
Another way. Look at just two opposite faces, say top and bottom. Together they contain all 8 vertices exactly once, so $2S = 1 + 2 + \cdots + 8 = 36$ and $S = 18$.

Answer: (C) 18

12 Geometry

Source: 2004 AMC 10B #12

An *annulus* is the region between two concentric circles. The concentric circles in the figure have radii b and c, with $b > c$. Let $\overline{OX}$ be a radius of the larger circle, let $\overline{XZ}$ be tangent to the smaller circle at Z, and let $\overline{OY}$ be the radius of the larger circle that contains Z. Let $a = XZ$, $d = YZ$, and $e = XY$. What is the area of the annulus?



(A) πa^2 **(B)** πb^2 **(C)** πc^2 **(D)** πd^2 **(E)** πe^2

Key idea. The area of the annulus is the big disk minus the small disk, $\pi b^2 - \pi c^2 = \pi(b^2 - c^2)$. So we want a length whose square is $b^2 - c^2$ — and a difference of two squares is exactly what the Pythagorean theorem produces.

Step 1. A tangent line is perpendicular to the radius drawn to the point of tangency. So $\overline{XZ} \perp \overline{OZ}$, and triangle OZX has a right angle at Z.

Step 2. In that right triangle the hypotenuse is $OX = b$ (a radius of the big circle) and the legs are $OZ = c$ (a radius of the small circle) and $XZ = a$. By the Pythagorean theorem,

$$a^2 + c^2 = b^2, \qquad \text{so} \qquad b^2 - c^2 = a^2.$$

Step 3. The area of the annulus is $\pi(b^2 - c^2) = \pi a^2$.

Notice that d and e were never needed. The figure labels them only to tempt you.

Watch out. πb^2 is the area of the whole large disk, hole included. And $d = b - c$ is the *width* of the ring, but $\pi d^2 = \pi(b - c)^2$ is not $\pi(b^2 - c^2)$: try $b = 5$, $c = 3$, where the ring has area 16π but $\pi d^2 = 4\pi$.

Answer: (A) πa^2

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13 Counting & Probability

Source: 2011 AMC 10A #13

How many even integers are there between 200 and 700 whose digits are all different and come from the set

$$\{1, 2, 5, 7, 8, 9\}?$$

(A) 12 **(B)** 20 **(C)** 72 **(D)** 120 **(E)** 200

Key idea. Fill the digits that have restrictions first — the hundreds digit (because of the range) and the units digit (because the number is even) — and leave the free tens digit for last.

Step 1 (hundreds digit). A three-digit number between 200 and 700 has hundreds digit $2, 3, 4, 5$ or 6. Only 2 and 5 are in the set.

Step 2 (units digit). The number is even, so the units digit must be an even digit from the set: 2 or 8. It also must differ from the hundreds digit.

- Hundreds digit 2: the units digit cannot be 2 again, so it must be 8 — 1 choice.
- Hundreds digit 5: the units digit can be 2 or 8 — 2 choices.

Step 3 (tens digit). Two of the six digits are already used, so the tens digit has 4 choices in every case.

Step 4. The total is $1 \cdot 4 + 2 \cdot 4 = 4 + 8 = 12$.

For the record, the twelve numbers are $218, 258, 278, 298, 512, 518, 528, 572, 578, 582, 592, 598$.

Answer: (A) 12

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14 Number Theory

Source: 2014 AMC 10B #14

> Danica drove her new car on a trip for a whole number of hours, averaging 55 miles per hour. At the beginning of the trip, abc miles was displayed on the odometer, where abc is a 3-digit number with $a \geq 1$ and $a + b + c \leq 7$. At the end of the trip, the odometer showed cba miles. What is $a^2 + b^2 + c^2$?
>
> **(A)** 26 **(B)** 27 **(C)** 36 **(D)** 37 **(E)** 41

Key idea. A number and its digit-reversal always differ by a multiple of 99, because the middle digit cancels. The distance driven is also a multiple of 55 — so look at what both facts force.

Step 1 (distance in digits). The distance driven is

$$cba - abc = (100c + 10b + a) - (100a + 10b + c) = 99c - 99a = 99(c - a).$$

Since the odometer went up, $c > a$.

Step 2 (use the 55). The trip lasted a whole number k of hours at 55 miles per hour, so $99(c-a) = 55k$. Dividing by 11: $9(c-a) = 5k$. Since 5 shares no factor with 9, the number $c-a$ must be a multiple of 5.

Step 3 (pin down the digits). We have $a \geq 1$ and $c \leq 9$, so $1 \leq c - a \leq 8$; the only multiple of 5 in that range is $c - a = 5$. Then $c = a + 5$, and $a + b + c \leq 7$ gives $2a + 5 + b \leq 7$, i.e. $2a + b \leq 2$. With $a \geq 1$ this forces $a = 1$, $b = 0$, and so $c = 6$.

Step 4. Check: the odometer went from 106 to 601, a distance of $495 = 55 \cdot 9$ miles, so the trip took 9 hours. Finally,

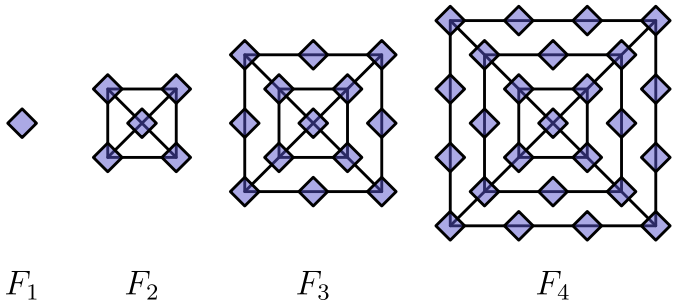
$$a^2 + b^2 + c^2 = 1 + 0 + 36 = 37.$$

Answer: (D) 37

15 Algebra

Source: 2009 AMC 10A #15

The figures F_1, F_2, F_3, and F_4 shown are the first in a sequence of figures. For $n \geq 3$, F_n is constructed from F_{n-1} by surrounding it with a square and placing one more diamond on each side of the new square than F_{n-1} had on each side of its outside square. For example, figure F_3 has 13 diamonds. How many diamonds are there in figure F_{20}?



F_1 F_2 F_3 F_4

(A) 401 **(B)** 485 **(C)** 585 **(D)** 626 **(E)** 761

Key idea. Twenty figures is too many to draw, so find how many diamonds each *new layer* adds, then add up the layers. Counting diamonds around a square is easiest if you count the corners carefully.

Step 1 (one layer). In F_n the outside square has n diamonds on each side, corners included (2 for F_2, 3 for F_3, 4 for F_4 in the picture). Adding 4 sides of n counts each of the 4 corners twice, so the outside square carries

$$4n - 4 = 4(n-1) \text{ diamonds.}$$

Check: $F_3 = F_2 + 4 \cdot 2 = 5 + 8 = 13$, as the problem says.

Step 2 (add the layers). Start from the single diamond of F_1 and add the squares for $n = 2, 3, \ldots, 20$:

$$F_{20} = 1 + 4(1 + 2 + \cdots + 19) = 1 + 4 \cdot \frac{19 \cdot 20}{2} = 1 + 4 \cdot 190 = 761.$$

Step 3 (general formula, optional). The same computation gives $F_n = 1 + 2n(n-1)$, which matches $F_1 = 1$, $F_2 = 5$, $F_3 = 13$, $F_4 = 25$.

Another way. Notice $1 + 2n(n-1) = n^2 + (n-1)^2$: the counts $1, 5, 13, 25, \ldots$ are $1^2 + 0^2$, $2^2 + 1^2$, $3^2 + 2^2$, $4^2 + 3^2$. So $F_{20} = 20^2 + 19^2 = 400 + 361 = 761$.

Watch out. Counting $4n$ diamonds on the new square (forgetting that corners are shared by two sides) gives $F_3 = 5 + 12 = 17$, which already contradicts the given 13 — always test your layer formula on the example the problem hands you.

Answer: (E) 761

16 Number Theory

Source: 2017 AMC 10B #16

> How many of the base-ten numerals for the positive integers less than or equal to 2017 contain the digit 0?
>
> **(A)** 469 **(B)** 471 **(C)** 475 **(D)** 478 **(E)** 481

Key idea. "Contains a 0" is awkward to count directly, but "contains *no* 0" is easy: every digit simply has 9 choices. So split the numbers by how many digits they have, and in each block count the complement.

Step 1. *One and two digits (*1–99*).* A one-digit number never contains 0. A two-digit number contains 0 only as its last digit, so it is one of $10, 20, \ldots, 90$: that is 9 numbers.

Step 2. *Three digits (*100–999*).* There are $999 - 100 + 1 = 900$ such numbers. Those with no 0 have each of the three digits chosen from 1–9, giving $9 \cdot 9 \cdot 9 = 729$. So $900 - 729 = 171$ contain a 0.

Step 3. *From* 1000 *to* 1999. There are 1000 such numbers. The first digit is 1; with no 0, each of the other three digits has 9 choices, giving 729. So $1000 - 729 = 271$ contain a 0.

Step 4. *From* 2000 *to* 2017. Every one of these has 0 as its second digit, and there are $2017 - 2000 + 1 = 18$ of them.

Step 5. Adding the blocks: $9 + 171 + 271 + 18 = 469$.

Watch out. Count *numbers*, not zeros. A number such as 100 or 2000 contains several zeros but is counted only once — complementary counting handles this automatically, while counting "zeros in the tens place, zeros in the units place, …" separately overcounts.

Answer: (A) 469

17 Geometry

Source: 2010 AMC 10A #17

A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid?

(A) 7 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 15

Key idea. Opposite faces share a hole, so there are really only three tunnels, one along each axis. They overlap in the middle, which signals *inclusion–exclusion* for the volume removed — or, even more simply, look directly at what is left.

Step 1. Each tunnel is a $2 \times 2 \times 3$ box, of volume 12. Three tunnels give $3 \cdot 12 = 36$, but this counts the overlaps more than once.

Step 2. Any two tunnels (say the vertical one and the front-to-back one) meet exactly in the central $2 \times 2 \times 2$ cube, of volume 8. All three tunnels also meet in that same central cube.

Step 3. By inclusion–exclusion, the volume removed is

$$36 - 3 \cdot 8 + 8 = 20.$$

Step 4. The remaining volume is $3^3 - 20 = 27 - 20 = 7$.

Watch out. Subtracting the three pairwise overlaps but forgetting to add the central cube back gives a removed volume of $36 - 24 = 12$ and a remainder of $27 - 12 = 15$, choice (E). The central cube lies in all three tunnels: it was added 3 times and subtracted 3 times, so it must be added once more.

Another way. Each 2-inch hole leaves a border of width $\frac{1}{2}$ inch on each side of a face. A point survives exactly when at most one of its three coordinates lies in the middle 2 inches. So what remains is a skeleton of the cube's edges: 8 corner cubes of side $\frac{1}{2}$, with total volume $8 \cdot \frac{1}{8} = 1$, joined by 12 bars of size $\frac{1}{2} \times \frac{1}{2} \times 2$, with total volume $12 \cdot \frac{1}{2} = 6$. The total is $1 + 6 = 7$.

Answer: (A) 7

18 Algebra

Source: 2008 AMC 10B #18

Bricklayer Brenda would take 9 hours to build a chimney alone, and bricklayer Brandon would take 10 hours to build it alone. When they work together, they talk a lot, and their combined output is decreased by 10 bricks per hour. Working together, they build the chimney in 5 hours. How many bricks are in the chimney?

(A) 500 **(B)** 900 **(C)** 950 **(D)** 1000 **(E)** 1900

Key idea. The penalty is measured in *bricks* per hour, not as a fraction of the job, so the usual "fraction of the chimney per hour" method cannot absorb it. Name the number of bricks n and write every rate in bricks per hour.

Step 1. Brenda lays the whole chimney in 9 hours, so her rate is $\frac{n}{9}$ bricks per hour. Brandon's rate is $\frac{n}{10}$ bricks per hour.

Step 2. Together, with the talking penalty, they lay $\frac{n}{9}+\frac{n}{10}-10$ bricks per hour. In 5 hours they lay all n bricks:

$$5\left(\frac{n}{9}+\frac{n}{10}-10\right)=n.$$

Step 3. Since $\frac{n}{9}+\frac{n}{10}=\frac{10n+9n}{90}=\frac{19n}{90}$, the equation becomes $\frac{95n}{90}-50=n$. Subtract $n=\frac{90n}{90}$ from both sides: $\frac{5n}{90}=50$, that is, $\frac{n}{18}=50$, so $n=900$.

Check. Brenda lays $900/9=100$ bricks per hour and Brandon $900/10=90$. Together that is 190, minus 10 for talking, so 180 bricks per hour, and $5\cdot 180=900$. ✓

Watch out. Read "decreased by 10 bricks per hour" as a rate. Subtracting 10 bricks once in total, $5\cdot\frac{19n}{90}-10=n$, gives $n=180$, which is not among the choices.

Answer: (B) 900

19 Number Theory

Source: 2013 AMC 10A #19

> In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b-representation of 2013 end in the digit 3?
>
> **(A)** 6 **(B)** 9 **(C)** 13 **(D)** 16 **(E)** 18

Key idea. The last digit of a number in base b is its remainder on division by b — just as the last decimal digit of 2013 is its remainder mod 10. So "ends in 3" turns into a divisibility condition, and counting bases becomes counting divisors.

Step 1. Writing $2013 = (\ldots d_1 d_0)_b$ means $2013 = b \cdot (\text{something}) + d_0$ with $0 \le d_0 \le b-1$. So the last digit d_0 is the remainder of 2013 divided by b. (Check with the example: $2013 = 9 \cdot 223 + 6$, and $(2676)_9$ ends in 6.)

Step 2. We need the remainder to be 3, so b divides $2013 - 3 = 2010$. Also the digit 3 must exist in base b, so $b \ge 4$. Conversely, if $b \mid 2010$ and $b \ge 4$, then $2013 = 2010 + 3$ leaves remainder 3, so it works.

Step 3. Factor: $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Each of the four primes is either in a divisor or not, so 2010 has $2^4 = 16$ divisors.

Step 4. The divisors below 4 are 1, 2, 3. Removing them leaves $16 - 3 = 13$ bases:

$$5,\ 6,\ 10,\ 15,\ 30,\ 67,\ 134,\ 201,\ 335,\ 402,\ 670,\ 1005,\ 2010.$$

For instance, $2013 = 1 \cdot 2010 + 3$, so in base 2010 the number has the two digits 1 and 3, and it indeed ends in 3.

Watch out. Counting all 16 divisors of 2010 gives choice (D). The bases 1, 2, 3 divide 2010, but they have no digit 3 — dividing 2013 by 2 or 3 leaves remainder 1 or 0, not 3.

Answer: (C) 13

20 Counting & Probability

Source: 2012 AMC 10A #20

A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random.

The square is then rotated $90°$ clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability the grid is now entirely black?

(A) $\frac{49}{512}$ **(B)** $\frac{7}{64}$ **(C)** $\frac{121}{1024}$ **(D)** $\frac{81}{512}$ **(E)** $\frac{9}{32}$

Key idea. A rotation by $90°$ never mixes corners with edges: the center stays put, the four corners cycle among themselves, and so do the four edge squares. So the grid splits into three independent pieces, and each piece can be analysed by asking exactly when a position ends up white.

Step 1. *When does a position end up white?* After the rotation, a position holds the square that was rotated into it. If that square is black, the position is black. If it is white, it is repainted black unless the position was *formerly* white too. So a position ends white exactly when both the square that moves into it and the square that was there before are white.

Step 2. *The center.* The center square rotates onto itself, so it ends white exactly when it started white. It must start black: probability $\frac{1}{2}$.

Step 3. *The corners.* Going clockwise, each corner moves into the next corner's position. By Step 1, the corners all end black exactly when no two *neighboring* corners in this cycle are both white — including the last and first, since the cycle wraps around. Of the $2^4 = 16$ colorings, the good ones are:

- no white: 1 coloring;
- one white: 4 colorings;
- two whites, diagonally opposite (not neighbors in the cycle): 2 colorings.

Three or more whites always include two neighbors. That makes $1+4+2=7$, so the probability is $\frac{7}{16}$.

Step 4. *The edge squares* form the same kind of 4-cycle, independently, so again the probability is $\frac{7}{16}$.

Step 5. Multiplying the three independent pieces:

$$\frac{1}{2} \cdot \frac{7}{16} \cdot \frac{7}{16} = \frac{49}{512}.$$

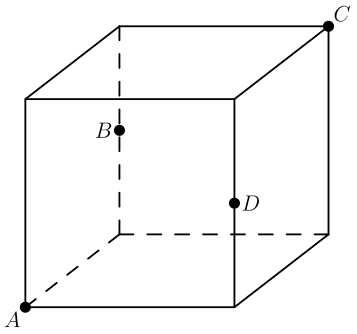
Watch out. The cycle wraps around: the corner that moves last lands where the first one was. Treating the four corners as a line instead of a circle wrongly allows two whites at the "ends". Forgetting the center square altogether gives $\frac{49}{256}$, which is not a choice.

Answer: (A) $\frac{49}{512}$

21 Geometry

Source: 2008 AMC 10A #21

A cube with side length 1 is sliced by a plane that passes through two diagonally opposite vertices A and C and the midpoints B and D of two opposite edges not containing A or C, as shown. What is the area of quadrilateral $ABCD$?



(A) $\frac{\sqrt{6}}{2}$ **(B)** $\frac{5}{4}$ **(C)** $\sqrt{2}$ **(D)** $\frac{3}{2}$ **(E)** $\sqrt{3}$

Key idea. The four sides of $ABCD$ all look alike — each runs from a vertex of the cube to the midpoint of an edge, across one face. Equal sides make $ABCD$ a rhombus, and a rhombus has an area formula that needs only its two diagonals, both of which are easy lengths in a cube.

Step 1. Put the cube on coordinates with $A = (0, 0, 0)$ and $C = (1, 1, 1)$, the x-axis pointing right, the y-axis pointing back and the z-axis up. Reading the figure, B is the midpoint of the back-left vertical edge and D is the midpoint of the front-right vertical edge:

$$B = \left(0, 1, \tfrac{1}{2}\right), \qquad D = \left(1, 0, \tfrac{1}{2}\right).$$

Step 2. Each side lies in one face of the cube and is the hypotenuse of a right triangle with legs 1 and $\frac{1}{2}$. For example, AB lies in the left face and goes 1 back and $\frac{1}{2}$ up. Similarly BC lies in the back face, CD in the right face, DA in the front face. So every side has length

$$\sqrt{1^2 + \left(\tfrac{1}{2}\right)^2} = \frac{\sqrt{5}}{2},$$

and $ABCD$, being a flat quadrilateral with four equal sides, is a rhombus.

Step 3. The diagonal AC is a space diagonal of the cube: $AC = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. The diagonal BD is horizontal at height $\frac{1}{2}$ and runs from $(0, 1)$ to $(1, 0)$ in the xy-directions, so it is as long as a face diagonal: $BD = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Step 4. The diagonals of a rhombus are perpendicular and cut it into four congruent right triangles, so its area is half the product of the diagonals:

$$\text{Area of } ABCD = \frac{1}{2} \cdot \sqrt{3} \cdot \sqrt{2} = \frac{\sqrt{6}}{2}.$$

Watch out. It is tempting to treat $ABCD$ as a square and square the side: $\left(\frac{\sqrt{5}}{2}\right)^2 = \frac{5}{4}$, choice (B). But its diagonals $\sqrt{3}$ and $\sqrt{2}$ are unequal, so it is not a square, and its area is less than side^2.

Answer: (A) $\frac{\sqrt{6}}{2}$

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22 Counting & Probability

Source: 2007 AMC 10B #22

A player chooses one of the numbers 1 through 4. After the choice has been made, two regular four-sided (tetrahedral) dice are rolled, with the sides of the dice numbered 1 through 4. If the number chosen appears on the bottom of exactly one die after it is rolled, then the player wins \$1. If the number chosen appears on the bottom of both of the dice, then the player wins \$2. If the number chosen does not appear on the bottom of either of the dice, the player loses \$1. What is the expected return to the player, in dollars, for one roll of the dice?

(A) $-\frac{1}{8}$ **(B)** $-\frac{1}{16}$ **(C)** 0 **(D)** $\frac{1}{16}$ **(E)** $\frac{1}{8}$

Key idea. "Expected return" means: for each outcome, multiply the money won by its probability, and add. All that matters is *how many* of the two dice show the chosen number — 0, 1 or 2 — so find those three probabilities and make sure they add to 1.

Step 1. Whatever number the player picks, each die shows it on the bottom with probability $\frac{1}{4}$ and misses it with probability $\frac{3}{4}$. The two dice are independent, and there are $4 \cdot 4 = 16$ equally likely outcomes.

Step 2. The three cases:

- both dice miss: $\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$;
- exactly one die hits: the first hits and the second misses, or the other way round, $2 \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{6}{16}$;
- both dice hit: $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

Check: $\frac{9}{16} + \frac{6}{16} + \frac{1}{16} = 1$.

Step 3. The expected return, in dollars, is

$$(-1) \cdot \frac{9}{16} + 1 \cdot \frac{6}{16} + 2 \cdot \frac{1}{16} = \frac{-9+6+2}{16} = -\frac{1}{16}.$$

Watch out. Taking "exactly one die" to have probability $\frac{1}{4} + \frac{1}{4} = \frac{8}{16}$ forgets that the *other* die must miss, so the double hit is counted here as well. It gives $\frac{-9+8+2}{16} = \frac{1}{16}$, choice (D). The warning sign: $\frac{9}{16} + \frac{8}{16} + \frac{1}{16}$ is more than 1.

Another way. Count outcomes instead of multiplying probabilities. Say the player picks 1 (by symmetry the choice does not matter). Of the 16 equally likely rolls, the dice avoid 1 in $3 \cdot 3 = 9$ rolls and both

show 1 in just 1 roll, so exactly one shows 1 in the remaining $16-9-1=6$ rolls. Over all 16 rolls the player's total is $9\cdot(-1)+6\cdot 1+1\cdot 2=-1$ dollar, an average of $-\frac{1}{16}$ per roll.

Answer: (B) $-\frac{1}{16}$

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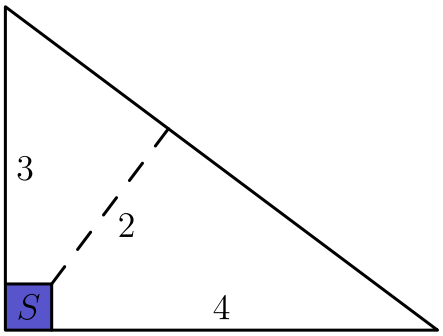
23 Geometry

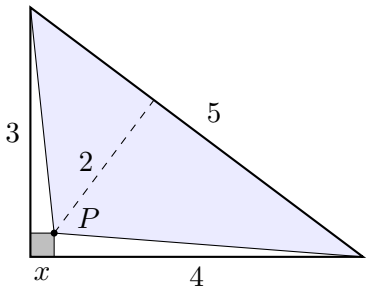
Source: 2018 AMC 10A #23

Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



(A) $\frac{25}{27}$ **(B)** $\frac{26}{27}$ **(C)** $\frac{73}{75}$ **(D)** $\frac{145}{147}$ **(E)** $\frac{74}{75}$

Key idea. A distance to a line is a *height*. The given "2" is the height of a triangle whose base is the hypotenuse (length 5) and whose apex is the outer corner P of the square. Joining P to the vertices cuts the field into pieces whose areas are all easy in terms of the square's side x, and the pieces must add up to the whole area 6.



Step 1. Let the square have side x, with the right angle at the origin, the leg of length 4 along the bottom and the leg of length 3 up the left side. The point of S closest to the hypotenuse is its outer corner $P=(x,x)$: the hypotenuse slants down to the right above the square, so moving left or down from P only takes you farther from it. So the dashed segment from P to the hypotenuse has length 2.

Step 2. Draw segments from P to the two acute vertices $(0,3)$ and $(4,0)$. Together with S, this splits the field into four pieces:

- the square S, of area x^2;
- the thin triangle on the left leg: base $3-x$ (from $(0,x)$ up to $(0,3)$), height x, area $\frac{x(3-x)}{2}$;
- the thin triangle on the bottom leg: base $4-x$, height x, area $\frac{x(4-x)}{2}$;
- the shaded triangle on the hypotenuse: base 5, height 2, area $\frac{5\cdot 2}{2}=5$.

Step 3. The whole field has area $\frac{3\cdot 4}{2}=6$, so

$$x^2+\frac{3x-x^2}{2}+\frac{4x-x^2}{2}+5=6.$$

The x^2 terms cancel: $x^2+\frac{7x-2x^2}{2}=\frac{7x}{2}$. So $\frac{7x}{2}=1$, and $x=\frac{2}{7}$.

Step 4. The square has area $\frac{4}{49}$, so the planted fraction is

$$\frac{6-\frac{4}{49}}{6}=1-\frac{4}{294}=1-\frac{2}{147}=\frac{145}{147}.$$

Watch out. The dashed segment is *parallel* to the altitude from the right angle, but it is not part of it. That altitude has length $\frac{3\cdot 4}{5}=\frac{12}{5}$; subtracting 2 from it, $\frac{12}{5}-2=\frac{2}{5}$, and using $\frac{2}{5}$ as the side of S gives $\frac{6-4/25}{6}=\frac{73}{75}$, choice (C).

Another way. With the same coordinates, the hypotenuse is the line $3X+4Y=12$, and the distance from (x,x) to it is $\frac{12-3x-4x}{5}=\frac{12-7x}{5}$. Setting this equal to 2 gives $7x=2$, so $x=\frac{2}{7}$ again.

Answer: (D) $\frac{145}{147}$

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24 Geometry

Source: 2011 AMC 10B #24

A lattice point in an xy-coordinate system is any point (x,y) where both x and y are integers. The graph of $y=mx+2$ passes through no lattice point with $0<x\le 100$ for all m such that $\frac{1}{2}<m<a$. What is the maximum possible value of a?

(A) $\frac{51}{101}$ **(B)** $\frac{50}{99}$ **(C)** $\frac{51}{100}$ **(D)** $\frac{52}{101}$ **(E)** $\frac{13}{25}$

Key idea. The "+2" is a decoy: since 2 is an integer, $mx+2$ is an integer exactly when mx is. So the line hits a lattice point above x exactly when $m=\frac{k}{x}$ for an integer k. The question becomes: *which fraction with denominator at most* 100 *lies closest above* $\frac{1}{2}$*?* The answer choices — all fractions just over $\frac{1}{2}$ with denominators near 100 — point the same way.

Step 1. For an integer x with $1\le x\le 100$, the point $(x,\,mx+2)$ is a lattice point exactly when mx is an integer k, that is, when $m=\frac{k}{x}$. So the bad slopes are exactly the fractions $\frac{k}{x}$ with $1\le x\le 100$, and a should be the smallest such fraction that is bigger than $\frac{1}{2}$.

Step 2. Fix x and find the smallest $\frac{k}{x} > \frac{1}{2}$, i.e. the smallest integer k with $2k > x$.

- x even, $x = 2j$: we need $k > j$, so $k \geq j+1$ and $\frac{k}{x} \geq \frac{j+1}{2j} = \frac{1}{2} + \frac{1}{x}$. The smallest of these over $x \leq 100$ is $\frac{1}{2} + \frac{1}{100} = \frac{51}{100}$.
- x odd, $x = 2j+1$: we need $k > j + \frac{1}{2}$, so $k \geq j+1$ and $\frac{k}{x} \geq \frac{j+1}{2j+1} = \frac{1}{2} + \frac{1}{2x}$. The smallest over odd $x \leq 99$ is $\frac{1}{2} + \frac{1}{198} = \frac{50}{99}$.

Since $\frac{1}{198} < \frac{1}{100}$, the smallest bad slope above $\frac{1}{2}$ is $\frac{50}{99}$.

Step 3. $a = \frac{50}{99}$ *works:* no bad slope lies strictly between $\frac{1}{2}$ and $\frac{50}{99}$, so for every such m the line misses all lattice points with $0 < x \leq 100$.

No larger a works: if $a > \frac{50}{99}$, then $m = \frac{50}{99}$ is allowed, and that line passes through the lattice point $(99,\ 50+2) = (99, 52)$.

So the maximum is $a = \frac{50}{99}$.

Watch out. It is natural to think the largest denominator, 100, gives the fraction closest to $\frac{1}{2}$, leading to $\frac{51}{100}$, choice (C). But even denominators can only get within $\frac{1}{x}$ of $\frac{1}{2}$, while odd ones get within $\frac{1}{2x}$, so $\frac{50}{99} \approx 0.50505$ beats $\frac{51}{100} = 0.51$. Choice (A), $\frac{51}{101}$, is smaller than $\frac{50}{99}$: it is a value of a that works, but not the largest.

Answer: (B) $\frac{50}{99}$

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25 Algebra

Source: 2010 AMC 10B #25

Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$P(1) = P(3) = P(5) = P(7) = a,$$

and

$$P(2) = P(4) = P(6) = P(8) = -a.$$

What is the smallest possible value of a?

(A) 105 **(B)** 315 **(C)** 945 **(D)** 7! **(E)** 8!

Key idea. When a polynomial takes the *same* value at several points, subtract that value: $P(x) - a$ is zero at $x = 1, 3, 5, 7$, so it factors. "Integer coefficients" together with "smallest possible a" signals a divisibility argument: plugging in $x = 2, 4, 6, 8$ will force a to be a multiple of something. Then we must show that multiple really occurs.

Step 1. Let $R(x) = (x-1)(x-3)(x-5)(x-7)$. Divide $P(x) - a$ by $R(x)$. Because R has leading coefficient 1 and integer coefficients, long division never creates fractions, so

$$P(x) - a = R(x)\,Q(x) + T(x)$$

with Q and T having integer coefficients and T of degree at most 3. Since $P(x)-a$ and $R(x)$ both vanish at $1,3,5,7$, so does T; a polynomial of degree at most 3 with four roots is 0. Hence $P(x) = a+R(x)Q(x)$.

Step 2. Evaluate R at the even points:

$$R(2) = (1)(-1)(-3)(-5) = -15, \quad R(4) = (3)(1)(-1)(-3) = 9,$$

$$R(6) = (5)(3)(1)(-1) = -15, \quad R(8) = (7)(5)(3)(1) = 105.$$

Step 3. At each even point $P = -a$, so $R(k)Q(k) = -2a$:

$$-15\,Q(2) = 9\,Q(4) = -15\,Q(6) = 105\,Q(8) = -2a.$$

Each $Q(k)$ is an integer, so $9 \mid 2a$ and $105 \mid 2a$ (and $15 \mid 2a$). Since 9 and 105 are odd, $9 \mid a$ and $105 = 3 \cdot 5 \cdot 7$ divides a. Hence a is a multiple of $\operatorname{lcm}(9,105) = 3^2 \cdot 5 \cdot 7 = 315$, and $a \geq 315$.

Step 4. *$a = 315$ is achievable.* We need an integer polynomial Q with

$$Q(2) = \tfrac{2a}{15} = 42, \quad Q(4) = -\tfrac{2a}{9} = -70, \quad Q(6) = 42, \quad Q(8) = -\tfrac{2a}{105} = -6.$$

Since $Q(2) = Q(6) = 42$, try $Q(x) = 42+(x-2)(x-6)L(x)$ with L linear. At $x = 4$: $42+(2)(-2)L(4) = -70$ gives $L(4) = 28$. At $x = 8$: $42 + (6)(2)L(8) = -6$ gives $L(8) = -4$. The line through these has slope $\frac{-4-28}{4} = -8$, so $L(x) = 60 - 8x$. Thus

$$P(x) = 315 + (x-1)(x-3)(x-5)(x-7)\big(42 + (x-2)(x-6)(60-8x)\big)$$

has integer coefficients, equals 315 at $1,3,5,7$, and equals -315 at $2,4,6,8$.

> **Watch out.** Using only $x = 2,6,8$ gives $\operatorname{lcm}(15,105) = 105$, choice (A); the condition from $x = 4$, where $R(4) = 9$, adds a second factor of 3. Multiplying $9 \cdot 105 = 945$ instead of taking the least common multiple gives choice (C) — but 315 is already divisible by both.

Answer: (B) 315