
AMC 10

SET 2

PRACTICE EXAMINATION

SOLUTIONS

[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 25 problems are taken from past AMC 10 contests (2000–2019) and appear in their original wording, rearranged into a new paper. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	C	2017 AMC 10A #1	Algebra
2	E	2014 AMC 10B #2	Algebra
3	D	2015 AMC 10A #3	Algebra
4	C	2008 AMC 10B #4	Algebra
5	D	2016 AMC 10A #5	Number Theory
6	C	2011 AMC 10A #6	Counting & Probability
7	B	2013 AMC 10B #7	Geometry
8	B	2004 AMC 10A #8	Counting & Probability
9	A	2009 AMC 10B #9	Geometry
10	B	2011 AMC 10B #10	Algebra
11	B	2015 AMC 10B #11	Counting & Probability
12	A	2017 AMC 10B #12	Algebra
13	A	2010 AMC 10A #13	Algebra
14	D	2016 AMC 10B #14	Geometry
15	D	2008 AMC 10A #15	Algebra
16	D	2009 AMC 10A #16	Algebra
17	C	2012 AMC 10A #17	Number Theory
18	E	2010 AMC 10B #18	Counting & Probability
19	C	2003 AMC 10A #19	Geometry
20	E	2006 AMC 10B #20	Geometry
21	C	2007 AMC 10A #21	Geometry
22	D	2005 AMC 10A #22	Number Theory
23	B	2006 AMC 10A #23	Geometry
24	B	2019 AMC 10A #24	Algebra
25	B	2014 AMC 10A #25	Number Theory

1 Algebra

Source: 2017 AMC 10A #1

What is the value of

$$2(2(2(2(2(2+1)+1)+1)+1)+1)+1?$$

- (A) 70 (B) 97 (C) 127 (D) 159 (E) 729

Key idea. Work from the inside out. Every layer does the same thing: double what is inside, then add 1.

The innermost bracket is $2 + 1 = 3$. Now apply “double and add 1” once for each remaining layer:

$$2 \cdot 3 + 1 = 7, \quad 2 \cdot 7 + 1 = 15, \quad 2 \cdot 15 + 1 = 31, \quad 2 \cdot 31 + 1 = 63, \quad 2 \cdot 63 + 1 = 127.$$

That uses all five outer layers, so the value is 127.

Notice the pattern: 3, 7, 15, 31, 63, 127 are each one less than a power of 2 ($4 - 1$, $8 - 1$, ..., $128 - 1$). Spotting it is a nice check on the arithmetic.

Watch out. Count the layers carefully. Stopping one layer early gives 63, and going one too far gives 255 — neither is a choice, which tells you a layer was miscounted.

Answer: (C) 127
2 Algebra

Source: 2014 AMC 10B #2

What is $\frac{2^3 + 2^3}{2^{-3} + 2^{-3}}$?

- (A) 16 (B) 24 (C) 32 (D) 48 (E) 64

Key idea. Both the top and the bottom are “something plus itself”, so factor out the 2 and cancel it.

The numerator is $2^3 + 2^3 = 2 \cdot 2^3$ and the denominator is $2^{-3} + 2^{-3} = 2 \cdot 2^{-3}$. The factors of 2 cancel:

$$\frac{2 \cdot 2^3}{2 \cdot 2^{-3}} = \frac{2^3}{2^{-3}} = 2^{3-(-3)} = 2^6 = 64.$$

As a check with plain numbers: the top is $8 + 8 = 16$, the bottom is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, and $16 \div \frac{1}{4} = 16 \cdot 4 = 64$.

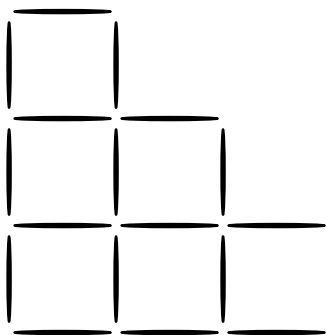
Watch out. When dividing powers, *subtract* the exponents, and subtracting -3 means adding 3. Writing $2^{3-3} = 2^0 = 1$ is the classic slip.

Answer: (E) 64

3 Algebra

Source: 2015 AMC 10A #3

Ann made a 3-step staircase using 18 toothpicks as shown in the figure. How many toothpicks does she need to add to complete a 5-step staircase?



- (A) 9 (B) 18 (C) 20 (D) 22 (E) 24

Key idea. Build the staircase one step at a time and count only the *new* toothpicks each step needs.

Each new step is a new bottom row with one more square than the row above it. Count what the k -th row needs:

- k horizontal toothpicks along its bottom;
- $k + 1$ vertical toothpicks (one more than the number of squares);
- 1 horizontal toothpick on top of its last square, since the row above is one square shorter and only covers the first $k - 1$ tops.

So row k adds $k + (k + 1) + 1 = 2k + 2$ toothpicks.

Check against the picture: rows 1, 2, 3 add 4, 6, 8, and $4 + 6 + 8 = 18$. Good.

To go from 3 steps to 5 steps Ann adds row 4 and row 5: $(2 \cdot 4 + 2) + (2 \cdot 5 + 2) = 10 + 12 = 22$.

Watch out. It is easy to forget the single toothpick on top of the new row's last square. Then each row seems to add only $2k + 1$, giving $9 + 11 = 20$, choice (C).

Answer: (D) 22

4 Algebra

Source: 2008 AMC 10B #4

A semipro baseball league has teams with 21 players each. League rules state that a player must be paid at least \$15,000, and that the total of all players' salaries for each team cannot exceed \$700,000. What is the maximum possible salary, in dollars, for a single player?

- (A) 270,000 (B) 385,000 (C) 400,000 (D) 430,000 (E) 700,000

Key idea. To make one salary as large as possible, make everyone else's as small as possible.

Every other player must earn at least \$15,000, so pay the other 20 players exactly that. Together they cost

$$20 \cdot \$15,000 = \$300,000.$$

That leaves at most $\$700,000 - \$300,000 = \$400,000$ for the star player.

This is actually reachable: 20 salaries of \$15,000 plus one of \$400,000 total exactly \$700,000, and everyone is paid at least the minimum. So the maximum is 400,000.

Watch out. The star is one of the 21 players. Subtracting the minimum for all 21 players gives $700,000 - 315,000 = 385,000$, choice (B).

Answer: (C) 400,000

5 Number Theory

Source: 2016 AMC 10A #5

A rectangular box has integer side lengths in the ratio 1 : 3 : 4. Which of the following could be the volume of the box?

- (A) 48 (B) 56 (C) 64 (D) 96 (E) 144

Key idea. Name the shortest side s . Then every possible volume has the same shape, $12s^3$.

If the shortest side is s , the sides are s , $3s$ and $4s$, with s a positive integer. The volume is

$$s \cdot 3s \cdot 4s = 12s^3.$$

So the volume must be 12 times a perfect cube. The smallest possibilities are

$$12 \cdot 1^3 = 12, \quad 12 \cdot 2^3 = 96, \quad 12 \cdot 3^3 = 324.$$

Only 96 appears among the choices; it is the box with sides 2, 6, 8.

Watch out. Being a multiple of 12 is not enough. $48 = 12 \cdot 4$ and $144 = 12 \cdot 12$ are multiples of 12, but 4 and 12 are not cubes, so no such box has those volumes.

Answer: (D) 96

6 Counting & Probability

Source: 2011 AMC 10A #6

Set A has 20 elements, and set B has 15 elements. What is the smallest possible number of elements in $A \cup B$, the union of A and B ?

- (A) 5 (B) 15 (C) 20 (D) 35 (E) 300

Key idea. The union is smallest when the two sets overlap as much as possible.

The union $A \cup B$ contains every element of A , so it has *at least* 20 elements, whatever B is.

Can it have exactly 20? Yes: choose B to be 15 of the 20 elements of A . Then B adds nothing new and $A \cup B = A$, which has 20 elements.

So the smallest possible size is 20. (In general $|A \cup B| = |A| + |B| - |A \cap B| = 35 - |A \cap B|$, and the overlap $|A \cap B|$ can be at most 15.)

Watch out. 35, choice (D), is the *largest* possible union (no overlap). And $5 = 20 - 15$, choice (A), is impossible: the union can never be smaller than A itself.

Answer: (C) 20

7 Geometry

Source: 2013 AMC 10B #7

Six points are equally spaced around a circle of radius 1. Three of these points are the vertices of a triangle that is neither equilateral nor isosceles. What is the area of this triangle?

- (A) $\frac{\sqrt{3}}{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{2}$ (E) 2

Key idea. The six points make a regular hexagon. Sort the possible triangles by how far apart (in steps around the circle) their vertices are.

Step 1. Going around the circle, the three vertices split the six steps into three gaps that add to 6. The only options are $1 + 1 + 4$, $2 + 2 + 2$ and $1 + 2 + 3$.

Step 2. A gap of 1 step gives a side of length 1 (a hexagon side), a gap of 2 or 4 steps gives $\sqrt{3}$, and a gap of 3 steps is a diameter, length 2. So $1 + 1 + 4$ gives sides 1, 1, $\sqrt{3}$ (isosceles) and $2 + 2 + 2$ is equilateral. The triangle must be $1 + 2 + 3$, with sides 1, $\sqrt{3}$, 2.

Step 3. The side of length 2 is a diameter, so the angle opposite it is a right angle. Check: $1^2 + (\sqrt{3})^2 = 4 = 2^2$. The legs are 1 and $\sqrt{3}$, so the area is

$$\frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2}.$$

Watch out. The legs are the two *shorter* sides. Using 1 and the diameter 2 as “base and height” gives $\frac{1}{2} \cdot 1 \cdot 2 = 1$, choice (C).

Answer: (B) $\frac{\sqrt{3}}{2}$

8 Counting & Probability

Source: 2004 AMC 10A #8

A game is played with tokens according to the following rule. In each round, the player with the most tokens gives one token to each of the other players and also places one token into a discard pile. The game ends when some player runs out of tokens. Players A , B , and C start with 15, 14, and 13 tokens, respectively. How many rounds will there be in the game?

- (A) 36 (B) 37 (C) 38 (D) 39 (E) 40

Key idea. Play a few rounds by hand and look for a repeating cycle.

Step 1. Write the counts as (A, B, C) . In each round the leader loses 3 tokens net (2 given away, 1 discarded) and the other two gain 1 each:

$$(15, 14, 13) \rightarrow (12, 15, 14) \rightarrow (13, 12, 15) \rightarrow (14, 13, 12).$$

After 3 rounds every player has exactly one token fewer, and the same order has returned.

Step 2. So every 3 rounds take $(n + 2, n + 1, n)$ to $(n + 1, n, n - 1)$. Starting from $n = 13$, after $3 \cdot 12 = 36$ rounds the counts are $(3, 2, 1)$ and nobody is out yet.

Step 3. In round 37, A has the most tokens and must give out 3: the counts become $(0, 3, 2)$. Player A has run out, so the game ends after 37 rounds.

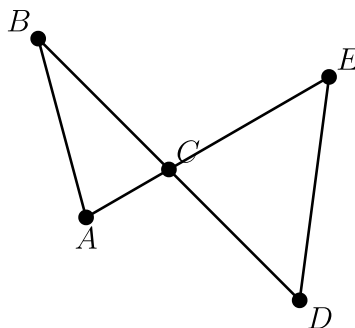
Watch out. “ C loses one token every 3 rounds, so C lasts $13 \cdot 3 = 39$ rounds” leads to choice (D). It ignores that the *leader* drops by 3 at once and can hit 0 first.

Answer: (B) 37

9 Geometry

Source: 2009 AMC 10B #9

Segment BD and AE intersect at C , as shown, $AB = BC = CD = CE$, and $\angle A = \frac{5}{2}\angle B$. What is the degree measure of $\angle D$?



- (A) 52.5 (B) 55 (C) 57.5 (D) 60 (E) 62.5

Key idea. The figure is two isosceles triangles joined at C . Solve the left one, then pass its angle at C across using vertical angles.

Step 1. In $\triangle ABC$, $AB = BC$, so the angles opposite these equal sides are equal: $\angle A = \angle ACB$. Let $\angle B = x$. Then $\angle A = \angle ACB = \frac{5}{2}x$, and

$$\frac{5}{2}x + x + \frac{5}{2}x = 180^\circ \implies 6x = 180^\circ \implies x = 30^\circ.$$

So $\angle ACB = \frac{5}{2} \cdot 30^\circ = 75^\circ$.

Step 2. $\angle DCE$ and $\angle ACB$ are vertical angles, so $\angle DCE = 75^\circ$.

Step 3. In $\triangle CDE$, $CD = CE$, so $\angle D = \angle E$. Hence $2\angle D + 75^\circ = 180^\circ$, giving $\angle D = 52.5^\circ$.

Watch out. Equal angles sit *opposite* equal sides. Since $AB = BC$, the equal angles are at A and C , not at B . Taking $\angle B = \angle C$ instead gives $\angle D = 70^\circ$, which is not a choice.

Answer: (A) 52.5

10 Algebra

Source: 2011 AMC 10B #10

Consider the set of numbers $\{1, 10, 10^2, 10^3, \dots, 10^{10}\}$. The ratio of the largest element of the set to the sum of the other ten elements of the set is closest to which integer?

- (A) 1 (B) 9 (C) 10 (D) 11 (E) 101

Key idea. Write the sum of the smaller powers of 10 as a string of 1's — then the ratio is easy to estimate.

The other ten elements are $1, 10, 10^2, \dots, 10^9$. Their sum is

$$S = 1 + 10 + 100 + \dots + 1,000,000,000 = 1,111,111,111.$$

Now compare with $10^{10} = 10,000,000,000$. Since $9 \cdot 1,111,111,111 = 9,999,999,999$, which is just 1 less than 10^{10} ,

$$\frac{10^{10}}{S} = \frac{9,999,999,999 + 1}{1,111,111,111} = 9 + \frac{1}{1,111,111,111},$$

which is a hair more than 9. The closest integer is 9.

Watch out. Comparing 10^{10} only with the next-largest element 10^9 gives 10, choice (C). The smaller terms add about 11% to the sum, which pulls the ratio down to 9.

Answer: (B) 9
11 Counting & Probability

Source: 2015 AMC 10B #11

Among the positive integers less than 100, each of whose digits is a prime number, one is selected at random. What is the probability that the selected number is prime?

- (A) $\frac{8}{99}$ (B) $\frac{2}{5}$ (C) $\frac{9}{20}$ (D) $\frac{1}{2}$ (E) $\frac{9}{16}$

Key idea. The number is chosen only from the integers whose digits are all prime, so first list that (small) sample space, then count the primes in it. The phrase “each of whose digits is a prime number” is what shrinks the pool — there are only four prime digits.

Step 1: the sample space. The prime digits are 2, 3, 5, 7. Numbers less than 100 have one or two digits.

- One digit: 2, 3, 5, 7 — that is 4 numbers.
- Two digits: 4 choices for the tens digit and 4 for the units digit, so $4 \cdot 4 = 16$ numbers.

In total there are $4 + 16 = 20$ equally likely choices.

Step 2: the primes. All four one-digit numbers 2, 3, 5, 7 are prime.

A two-digit number ending in 2 is even and one ending in 5 is a multiple of 5, so neither is prime. That leaves the eight numbers ending in 3 or 7:

$$23, 33, 53, 73, \quad 27, 37, 57, 77.$$

Of these, $33 = 3 \cdot 11$, $27 = 3 \cdot 9$, $57 = 3 \cdot 19$ and $77 = 7 \cdot 11$ are composite, while 23, 53, 73, 37 are prime.

Step 3. So $4 + 4 = 8$ of the 20 choices are prime, and the probability is

$$\frac{8}{20} = \frac{2}{5}.$$

Watch out. The number is *not* chosen from all of $1, \dots, 99$. Dividing the 8 primes by 99 gives $\frac{8}{99}$, choice (A).

Watch out. 57 looks prime but is $3 \cdot 19$. Counting it would give $\frac{9}{20}$, choice (C).

Answer: (B) $\frac{2}{5}$

12 Algebra

Source: 2017 AMC 10B #12

Elmer's new car gets 50% better fuel efficiency, measured in kilometers per liter, than his old car. However, his new car uses diesel fuel, which is 20% more expensive per liter than the gasoline his old car uses. By what percent will Elmer save money if he uses his new car instead of his old car for a long trip?

- (A) 20% (B) $26\frac{2}{3}\%$ (C) $27\frac{7}{9}\%$ (D) $33\frac{1}{3}\%$ (E) $41\frac{2}{3}\%$

Key idea. Percent changes like these multiply, they do not add. Since no actual numbers are given, pick convenient ones and compute the cost of the same trip in both cars.

Step 1: choose numbers. Say the old car gets 10 km per liter and gasoline costs \$1 per liter. Then the new car gets 50% more, 15 km per liter, and diesel costs 20% more, \$1.20 per liter. Take a trip of 150 km, which divides evenly.

Step 2: cost in each car. Old car: $150 \div 10 = 15$ liters, costing $15 \cdot \$1 = \15 . New car: $150 \div 15 = 10$ liters, costing $10 \cdot \$1.20 = \12 .

Step 3. Elmer saves $\$15 - \$12 = \$3$ out of \$15, which is $\frac{3}{15} = \frac{1}{5} = 20\%$.

In general: the new car uses $\frac{1}{1.5} = \frac{2}{3}$ as much fuel, at 1.2 times the price, so it costs $\frac{2}{3} \cdot 1.2 = 0.8$ times as much — a 20% saving, whatever trip length or prices you choose.

Watch out. Forgetting that diesel costs more: the new car uses $\frac{2}{3}$ as much fuel, a saving of $33\frac{1}{3}\%$, choice (D).

Watch out. “50% better minus 20% more expensive is 30%” is not how percents combine (and 30% is not a choice).

Answer: (A) 20%

13 Algebra

Source: 2010 AMC 10A #13

Angelina drove at an average rate of 80 kph and then stopped 20 minutes for gas. After the stop, she drove at an average rate of 100 kph. Altogether she drove 250 km in a total trip time of 3 hours including the stop. Which equation could be used to solve for the time t in hours that she drove before her stop?

- (A) $80t + 100\left(\frac{8}{3} - t\right) = 250$ (B) $80t = 250$ (C) $100t = 250$
 (D) $90t = 250$ (E) $80\left(\frac{8}{3} - t\right) + 100t = 250$

Key idea. Distance = rate $\times$ time for each leg of the trip, and the two distances add to 250. The only subtlety is that the 3 hours includes 20 minutes of *not* driving.

Step 1: driving time. The stop lasts 20 minutes = $\frac{1}{3}$ hour. So the total time spent driving is

$$3 - \frac{1}{3} = \frac{8}{3} \text{ hours.}$$

Step 2: each leg. Before the stop she drives for t hours at 80 kph, covering $80t$ km. After the stop she drives for the rest of the driving time, $\frac{8}{3} - t$ hours, at 100 kph, covering $100\left(\frac{8}{3} - t\right)$ km.

Step 3. The two legs total 250 km:

$$80t + 100\left(\frac{8}{3} - t\right) = 250.$$

That is choice (A). (Solving it: $80t + \frac{800}{3} - 100t = 250$, so $20t = \frac{50}{3}$ and $t = \frac{5}{6}$ hour, i.e. 50 minutes. Check: $80 \cdot \frac{5}{6} + 100 \cdot \frac{11}{6} = \frac{400}{6} + \frac{1100}{6} = 250$.)

Watch out. Choice (E) has the two speeds swapped: it assigns 80 kph to the time *after* the stop. Read which speed goes with t — t is the time *before* the stop.

Watch out. Forgetting to remove the 20-minute stop gives $80t + 100(3 - t) = 250$, which is not offered; if you wrote that, look for the $\frac{8}{3}$.

Answer: (A) $80t + 100\left(\frac{8}{3} - t\right) = 250$

14 Geometry

Source: 2016 AMC 10B #14

How many squares whose sides are parallel to the axes and whose vertices have coordinates that are integers lie entirely within the region bounded by the line $y = \pi x$, the line $y = -0.1$ and the line $x = 5.1$?

- (A) 30 (B) 41 (C) 45 (D) 50 (E) 57

Key idea. Count square by square, sorted by side length. The odd numbers -0.1 , 5.1 and π are there to tell you exactly which lattice points are allowed: $y \geq 0$, $x \leq 5$, and below the line the height at $x = a$ can be at most $3a$, because $\pi \approx 3.14$.

Step 1: which corner matters. The region is a triangle: above $y = -0.1$, left of $x = 5.1$, and below the rising line $y = \pi x$. For a square with lower-left corner (a, b) and side s , all its corners must be inside:

- the bottom side needs $b \geq 0$;
- the right side needs $a + s \leq 5$;
- the line rises to the right, so the corner closest to it is the *top-left* one, $(a, b+s)$. It needs $b+s < \pi a$.

Step 2: the line at integer x . At $x = 1, 2, 3, 4$ the line is at height $\pi, 2\pi, 3\pi, 4\pi \approx 3.14, 6.28, 9.42, 12.57$. Each lies between $3a$ and $3a + 1$, so for an integer height the condition $b + s < \pi a$ means exactly $b + s \leq 3a$. (At $a = 0$ the line is at height 0, so no square can have its left side on the y -axis.)

Step 3: count. For each left edge a , the bottom b can be $0, 1, \dots, 3a - s$, which is $3a - s + 1$ choices.

- $s = 1$, $a = 1, 2, 3, 4$: $3 + 6 + 9 + 12 = 30$.
- $s = 2$, $a = 1, 2, 3$: $2 + 5 + 8 = 15$.
- $s = 3$, $a = 1, 2$: $1 + 4 = 5$.
- $s = 4$: only $a = 1$ fits horizontally, but then the top would reach $4 > 3$. None.

The total is $30 + 15 + 5 = 50$.

Watch out. Counting only the unit squares gives 30, choice (A). The question asks for all squares, of every size.

Answer: (D) 50

15 Algebra

Source: 2008 AMC 10A #15

Yesterday Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?

- (A) 120 (B) 130 (C) 140 (D) 150 (E) 160

Key idea. Everything is described relative to Ian, so give Ian a speed r and time t . There are two unknowns but only one equation — a sign that you will not need r and t separately, only a combination of them.

Step 1: Han. Ian drove rt miles. Han drove at $r + 5$ for $t + 1$ hours, so his extra distance is

$$(r + 5)(t + 1) - rt = rt + r + 5t + 5 - rt = r + 5t + 5.$$

This equals 70, so $r + 5t = 65$.

Step 2: Jan. Jan drove at $r + 10$ for $t + 2$ hours, so his extra distance is

$$(r + 10)(t + 2) - rt = 2r + 10t + 20 = 2(r + 5t) + 20.$$

Step 3. Substitute $r + 5t = 65$: Jan drove $2 \cdot 65 + 20 = 150$ more miles than Ian.

Check with numbers: if Ian drove 5 hours at 40 mph (200 miles), Han drove 6 hours at 45 mph (270 miles, 70 more) and Jan drove 7 hours at 50 mph (350 miles, 150 more).

Watch out. “Jan’s extras are twice Han’s, so twice 70” gives 140, choice (C). Doubling both the extra time and the extra speed makes the “extra $\times$ extra” term $10 \cdot 2 = 20$, four times Han’s $5 \cdot 1 = 5$, not twice.

Answer: (D) 150

16 Algebra

Source: 2009 AMC 10A #16

Let a , b , c , and d be real numbers with $|a - b| = 2$, $|b - c| = 3$, and $|c - d| = 4$. What is the sum of all possible values of $|a - d|$?

- (A) 9 (B) 12 (C) 15 (D) 18 (E) 24

Key idea. The three given differences *chain together*: $a - b$, $b - c$, $c - d$ add up to $a - d$, because the middle letters cancel. Each absolute value only hides a sign, so list the sign choices.

Step 1. Write

$$a - d = (a - b) + (b - c) + (c - d).$$

From the conditions, $a - b = \pm 2$, $b - c = \pm 3$, and $c - d = \pm 4$, and each sign can be chosen freely (pick any a , then choose b , then c , then d). So $a - d = \pm 2 \pm 3 \pm 4$.

Step 2. Flipping all three signs only changes the sign of $a - d$, so it does not change $|a - d|$. We may therefore keep the $+$ on the 4 and try the four sign patterns for 2 and 3:

$$2 + 3 + 4 = 9, \quad -2 + 3 + 4 = 5, \quad 2 - 3 + 4 = 3, \quad -2 - 3 + 4 = -1.$$

The possible values of $|a - d|$ are 9, 5, 3, and 1.

Step 3. Their sum is $9 + 5 + 3 + 1 = 18$.

For example, $a = 0$, $b = 2$, $c = -1$, $d = 3$ satisfies all three conditions and gives $|a - d| = 3$; similar choices realise the other three values.

Watch out. Reading “all possible values” as “the largest value” and answering $2 + 3 + 4 = 9$, choice (A). The question wants every value that can occur, including the small ones where the steps partly cancel.

Answer: (D) 18

17 Number Theory

Source: 2012 AMC 10A #17

Let a and b be relatively prime positive integers with $a > b > 0$ and

$$\frac{a^3 - b^3}{(a - b)^3} = \frac{73}{3}.$$

What is $a - b$?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Key idea. The numerator $a^3 - b^3$ has a factor $a - b$, and after cancelling it what remains, $a^2 + ab + b^2$, should be compared with $(a - b)^2$. Since the question asks for $a - b$ itself, rewrite everything in terms of $a - b$ and ab .

Step 1. Factor the difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. Cancelling one factor $a - b$ (allowed, since $a > b$) gives

$$\frac{a^2 + ab + b^2}{(a - b)^2} = \frac{73}{3}.$$

Step 2. Because $(a - b)^2 = a^2 - 2ab + b^2$, the numerator is $a^2 + ab + b^2 = (a - b)^2 + 3ab$. So

$$1 + \frac{3ab}{(a - b)^2} = \frac{73}{3}, \quad \text{so} \quad \frac{3ab}{(a - b)^2} = \frac{70}{3},$$

that is, $9ab = 70(a - b)^2$.

Step 3. Now use “relatively prime”. If a prime p divided both $a - b$ and ab , it would divide a or b , and then (through $a - b$) the other one too — impossible. So $(a - b)^2$ shares no factor with ab , and since it divides $9ab$, it must divide 9. Hence $a - b = 1$ or $a - b = 3$.

Step 4. If $a - b = 1$, then $9ab = 70$, which has no integer solution. If $a - b = 3$, then $9ab = 630$, so $ab = 70$; with $a = b + 3$ this is $b^2 + 3b - 70 = 0$, i.e. $(b + 10)(b - 7) = 0$, so $b = 7$ and $a = 10$.

Check: $a^2 + ab + b^2 = 100 + 70 + 49 = 219$ and $(a - b)^2 = 9$, and $\frac{219}{9} = \frac{73}{3}$. So $a - b = 3$.

Another way. Cross-multiplying $3(a^2 + ab + b^2) = 73(a^2 - 2ab + b^2)$ gives $70a^2 - 149ab + 70b^2 = 0$, which factors as $(10a - 7b)(7a - 10b) = 0$. Since $a > b$, $\frac{a}{b} = \frac{10}{7}$, and relative primality forces $a = 10$, $b = 7$.

Answer: (C) 3

18 Counting & Probability

Source: 2010 AMC 10B #18

Positive integers a , b , and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \dots, 2010\}$.

What is the probability that $abc + ab + a$ is divisible by 3?

- (A) $\frac{1}{3}$ (B) $\frac{29}{81}$ (C) $\frac{31}{81}$ (D) $\frac{11}{27}$ (E) $\frac{13}{27}$

Key idea. Divisibility by 3 depends only on remainders mod 3, and $2010 = 3 \cdot 670$ splits evenly into the three remainder classes. Every term contains a , so factor it out and split into “ a is a multiple of 3” and “it is not”.

Step 1. Since 2010 is a multiple of 3, exactly 670 numbers in the set leave each remainder 0, 1, 2. So the remainders of a , b , c are independent and each is 0, 1, or 2 with probability $\frac{1}{3}$.

Step 2. Factor: $abc + ab + a = a(bc + b + 1)$. Because 3 is prime, this product is divisible by 3 exactly when $3 \mid a$ or $3 \mid bc + b + 1$.

Step 3. Case $3 \mid a$: probability $\frac{1}{3}$, and then the expression is divisible by 3 whatever b and c are.

Step 4. Case $3 \nmid a$ (probability $\frac{2}{3}$): we need $bc + b + 1 = b(c + 1) + 1 \equiv 0$, i.e. $b(c + 1) \equiv 2 \pmod{3}$. A product of remainders is 2 only as $1 \cdot 2$ or $2 \cdot 1$:

- $b \equiv 1$ and $c + 1 \equiv 2$, i.e. $c \equiv 1$;
- $b \equiv 2$ and $c + 1 \equiv 1$, i.e. $c \equiv 0$.

Each pair of remainders has probability $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$, so this case succeeds with probability $\frac{2}{9}$.

Step 5. Combine the cases:

$$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{2}{9} = \frac{9}{27} + \frac{4}{27} = \frac{13}{27}.$$

A direct check of all 27 remainder triples (a, b, c) confirms that exactly 13 of them work.

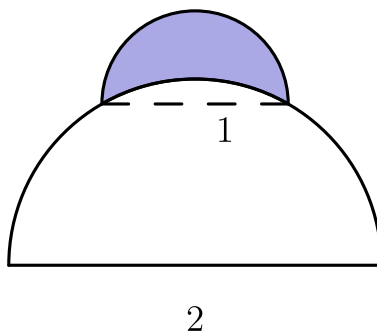
Watch out. Stopping after the obvious case “ a is a multiple of 3” gives $\frac{1}{3}$, choice (A). The factor $bc + b + 1$ can also be a multiple of 3 when a is not.

Answer: (E) $\frac{13}{27}$

19 Geometry

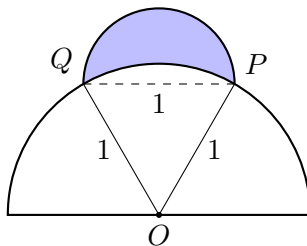
Source: 2003 AMC 10A #19

A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a *lune*. Determine the area of this lune.



- (A) $\frac{1}{6}\pi - \frac{\sqrt{3}}{4}$ (B) $\frac{\sqrt{3}}{4} - \frac{1}{12}\pi$ (C) $\frac{\sqrt{3}}{4} - \frac{1}{24}\pi$ (D) $\frac{\sqrt{3}}{4} + \frac{1}{24}\pi$ (E) $\frac{\sqrt{3}}{4} + \frac{1}{12}\pi$

Key idea. A lune is bounded by two arcs, so its area is a difference of round pieces. Take the whole small semicircle and remove the sliver of the big circle that pokes into it. That sliver is a circular segment, which you find as “sector minus triangle” — so join the chord’s ends to the big circle’s centre.



Step 1. Let O be the centre of the big semicircle (radius 1) and let P, Q be the ends of the small semicircle’s diameter, so $PQ = 1$. Both P and Q lie on the big circle, so $OP = OQ = 1$. Triangle OPQ is equilateral with side 1, its area is $\frac{\sqrt{3}}{4}$, and $\angle POQ = 60^\circ$.

Step 2. The part of the big disk above chord PQ is a segment:

$$\text{segment} = \text{sector } POQ - \triangle OPQ = \frac{60}{360}\pi(1)^2 - \frac{\sqrt{3}}{4} = \frac{\pi}{6} - \frac{\sqrt{3}}{4}.$$

Step 3. The small semicircle has radius $\frac{1}{2}$, so its area is $\frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$. The lune is this semicircle with the segment removed:

$$\frac{\pi}{8} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{4} + \frac{3\pi}{24} - \frac{4\pi}{24} = \frac{\sqrt{3}}{4} - \frac{1}{24}\pi.$$

Numerically this is about 0.302, a sensible size for a sliver inside a half-disk of area $\frac{\pi}{8} \approx 0.393$.

Watch out. Using the area of the whole circle of diameter 1, $\frac{\pi}{4}$, instead of the semicircle gives $\frac{\pi}{4} - \frac{\pi}{6} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} + \frac{1}{12}\pi$, choice (E). Only half of that circle is drawn.

Answer: (C) $\frac{\sqrt{3}}{4} - \frac{1}{24}\pi$

20 Geometry

Source: 2006 AMC 10B #20

In rectangle $ABCD$, we have $A = (6, -22)$, $B = (2006, 178)$, and $D = (8, y)$ for some integer y . What is the area of rectangle $ABCD$?

- (A) 4000 (B) 4040 (C) 4400 (D) 40,000 (E) 40,400

Key idea. In a rectangle the sides at a corner are perpendicular. A and B are known, D has one unknown coordinate, and AB and AD meet at A — so the right angle at A is exactly the one equation needed to find y . Then area = $AB \cdot AD$.

Step 1. Slope of $\overline{AB}$:

$$\frac{178 - (-22)}{2006 - 6} = \frac{200}{2000} = \frac{1}{10}.$$

Since $\overline{AD} \perp \overline{AB}$, the slope of $\overline{AD}$ is the negative reciprocal, -10 .

Step 2. Using $A = (6, -22)$ and $D = (8, y)$:

$$\frac{y - (-22)}{8 - 6} = -10 \implies y + 22 = -20 \implies y = -42.$$

So $D = (8, -42)$.

Step 3. Side lengths, by the distance formula:

$$AB = \sqrt{2000^2 + 200^2} = \sqrt{200^2(10^2 + 1)} = 200\sqrt{101},$$

$$AD = \sqrt{2^2 + (-20)^2} = \sqrt{404} = \sqrt{2^2 \cdot 101} = 2\sqrt{101}.$$

Step 4. Area = $AB \cdot AD = 200\sqrt{101} \cdot 2\sqrt{101} = 400 \cdot 101 = 40,400$.

Watch out. The rectangle is tilted only slightly, so it is tempting to call the sides “about 2000” and “about 20” and answer $2000 \cdot 20 = 40,000$, choice (D). The exact lengths are $200\sqrt{101}$ and $2\sqrt{101}$, so the area is $400 \cdot 101$, not $400 \cdot 100$.

Another way. Write the sides from A as vectors: $\overrightarrow{AB} = (2000, 200)$ and $\overrightarrow{AD} = (2, y + 22)$. Perpendicular means the dot product is 0: $4000 + 200(y + 22) = 0$, so $y + 22 = -20$. The area of the parallelogram (here a rectangle) spanned by $(2000, 200)$ and $(2, -20)$ is $|2000 \cdot (-20) - 200 \cdot 2| = |-40,000 - 400| = 40,400$.

Answer: (E) 40,400

21 Geometry

Source: 2007 AMC 10A #21

A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

- (A) 3 (B) 6 (C) 8 (D) 9 (E) 12

Key idea. Everything is linked through the sphere's *diameter*. “Inscribed in the cube” means the sphere touches the faces, so its diameter is the outer cube's edge. “Cube inscribed in the sphere” means the inner cube's corners lie on the sphere, so the diameter is the inner cube's *space* diagonal. Two inscriptions, two different lengths of the cube — that is why you must name which one each time.

Step 1: the outer cube. A cube has 6 congruent square faces, so each face has area $24 \div 6 = 4$ and the edge is $\sqrt{4} = 2$ meters.

Step 2: the sphere. The inscribed sphere touches all six faces, in particular two opposite faces, which are 2 meters apart. So the sphere's diameter is 2.

Step 3: the inner cube. The inner cube's centre is the sphere's centre, and all 8 of its corners lie on the sphere. Two opposite corners (the ends of a space diagonal) are symmetric through the centre, so the segment joining them is a diameter of the sphere. Hence the space diagonal of the inner cube is 2.

If the inner cube has edge s , its face diagonal is $s\sqrt{2}$, and the space diagonal is the hypotenuse of a right triangle with legs s and $s\sqrt{2}$:

$$\sqrt{s^2 + 2s^2} = s\sqrt{3} = 2, \quad \text{so} \quad s^2 = \frac{4}{3}.$$

Step 4: surface area. $6s^2 = 6 \cdot \frac{4}{3} = 8$ square meters.

Watch out. Setting the sphere's diameter equal to the inner cube's *face* diagonal gives $s\sqrt{2} = 2$, $s^2 = 2$, and surface area 12, choice (E). The corners that touch the sphere are opposite corners of the whole cube, not of one face.

Another way. Compare the two cubes directly. The outer cube's space diagonal is $2\sqrt{3}$, the inner cube's is 2, so the inner cube is the outer cube scaled by $\frac{1}{\sqrt{3}}$. Areas scale by the square, $\frac{1}{3}$, giving $24 \cdot \frac{1}{3} = 8$.

Answer: (C) 8

22 Number Theory

Source: 2005 AMC 10A #22

Let S be the set of the 2005 smallest positive multiples of 4, and let T be the set of the 2005 smallest positive multiples of 6. How many elements are common to S and T ?

- (A) 166 (B) 333 (C) 500 (D) 668 (E) 1001

Key idea. A number common to S and T must be a multiple of both 4 and 6, i.e. of their least common multiple. The only other thing that matters is where each list *stops*, since both sets are finite — so write down the largest element of each.

Step 1: what the common elements look like. A positive integer is a multiple of both $4 = 2^2$ and $6 = 2 \cdot 3$ exactly when it is a multiple of $\text{lcm}(4, 6) = 2^2 \cdot 3 = 12$.

Step 2: where the lists stop. $S = \{4, 8, \dots, 4 \cdot 2005\}$, so S consists of *all* positive multiples of 4 up to 8020. Similarly T consists of all positive multiples of 6 up to $6 \cdot 2005 = 12,030$.

Step 3: combine. A number lies in both sets exactly when it is a multiple of 12, is at most 8020, and is at most 12,030. The second condition already implies the third, so the shorter list S sets the cutoff.

Step 4: count. The multiples of 12 up to 8020 are $12 \cdot 1, 12 \cdot 2, \dots, 12 \cdot k$ with $12k \leq 8020$. Since $12 \cdot 668 = 8016 \leq 8020$ and $12 \cdot 669 = 8028 > 8020$, there are 668 of them.

Check with a different view: in $S = \{4 \cdot 1, \dots, 4 \cdot 2005\}$, the element $4m$ is a multiple of 12 exactly when $3 \mid m$, and there are $\lfloor 2005/3 \rfloor = 668$ such m between 1 and 2005.

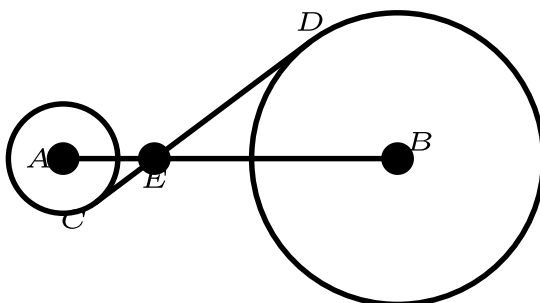
Watch out. Using T 's largest element 12,030 as the cutoff counts 1002 multiples of 12, which is too many: multiples of 12 beyond 8020 are in T but not in S . Also, the common multiples come from $\text{lcm}(4, 6) = 12$, not $4 \cdot 6 = 24$.

Answer: (D) 668

23 Geometry

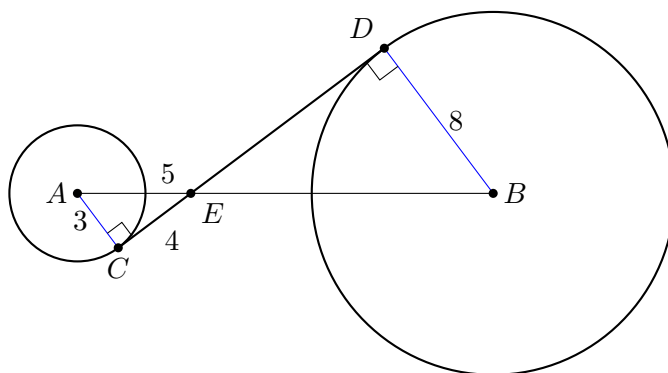
Source: 2006 AMC 10A #23

Circles with centers A and B have radii 3 and 8, respectively. A common internal tangent touches the circles at C and D , as shown. Lines AB and CD intersect at E , and $AE = 5$. What is CD ?



- (A) 13 (B) $\frac{44}{3}$ (C) $\sqrt{221}$ (D) $\sqrt{255}$ (E) $\frac{55}{3}$

Key idea. Whenever a tangent line appears, draw the radius to the point of tangency: it is perpendicular to the tangent. Here that creates two right triangles, ACE and BDE , whose angles at E are vertical angles — and the given $AE = 5$ with $AC = 3$ is a 3–4–5 triangle waiting to be finished.



Step 1: the small triangle. The radius $\overline{AC}$ is perpendicular to the tangent line CD , so $\triangle ACE$ has a right angle at C . Its hypotenuse is $AE = 5$ and one leg is $AC = 3$, so

$$CE = \sqrt{5^2 - 3^2} = \sqrt{16} = 4.$$

Step 2: the large triangle. Likewise $\overline{BD} \perp \overline{CD}$, so $\triangle BDE$ has a right angle at D . The angles at E are vertical angles (E lies on both lines AB and CD , between the circles), so they are equal. Two angles match, hence $\triangle ACE \sim \triangle BDE$, with $C \leftrightarrow D$.

Step 3: scale up. The ratio of the triangles is $\frac{BD}{AC} = \frac{8}{3}$, so

$$DE = \frac{8}{3} \cdot CE = \frac{8}{3} \cdot 4 = \frac{32}{3}.$$

Step 4: add. E lies between C and D on the tangent, so

$$CD = CE + DE = 4 + \frac{32}{3} = \frac{12}{3} + \frac{32}{3} = \frac{44}{3}.$$

Watch out. The same scale factor gives $BE = \frac{8}{3} \cdot 5 = \frac{40}{3}$, and adding $AE + BE = 5 + \frac{40}{3} = \frac{55}{3}$ produces choice (E). But that is AB , the distance between the *centres*, not the tangent length CD .

Another way. Slide $\overline{CD}$ parallel to itself so that C moves to A . Its other end lands on the extension of $\overline{BD}$ beyond D , at distance $8 + 3 = 11$ from B , which makes a right triangle with hypotenuse AB and legs CD and 11. This gives the internal-tangent formula $CD^2 = AB^2 - (3 + 8)^2$. With $AB = \frac{55}{3}$ (from the trap above, used correctly):

$$CD^2 = \frac{3025}{9} - \frac{1089}{9} = \frac{1936}{9}, \quad CD = \frac{44}{3}.$$

Answer: (B) $\frac{44}{3}$

24 Algebra

Source: 2019 AMC 10A #24

Let p , q , and r be the distinct roots of the polynomial

$$x^3 - 22x^2 + 80x - 67.$$

There exist real numbers A , B , and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s - p} + \frac{B}{s - q} + \frac{C}{s - r}$$

for all real numbers s with $s \notin \{p, q, r\}$. What is

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C}?$$

(A) 243

(B) 244

(C) 245

(D) 246

(E) 247

Key idea. You cannot find p , q , r exactly, so aim for an expression that is *symmetric* in them — then Vieta's formulas finish it. To get at A alone, clear the denominators and plug in $s = p$: the B and C terms vanish (the “cover-up” trick for partial fractions).

Step 1: clear denominators. The cubic factors as $s^3 - 22s^2 + 80s - 67 = (s - p)(s - q)(s - r)$. Multiplying the identity by this product gives

$$1 = A(s - q)(s - r) + B(s - p)(s - r) + C(s - p)(s - q).$$

Both sides are polynomials that agree for all s except three values, so they agree for every s — including $s = p, q, r$.

Step 2: plug in the roots. Setting $s = p$ kills the B and C terms: $1 = A(p - q)(p - r)$. Doing the same with $s = q$ and $s = r$:

$$\frac{1}{A} = (p - q)(p - r), \quad \frac{1}{B} = (q - p)(q - r), \quad \frac{1}{C} = (r - p)(r - q).$$

Step 3: expand and add. Each product expands as, for example, $(p - q)(p - r) = p^2 - pq - pr + qr$. Adding all three, every square appears once; each product such as pq appears with a minus sign twice (from the p and q terms) and with a plus sign once (from the r term). So

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = p^2 + q^2 + r^2 - (pq + qr + rp).$$

Step 4: Vieta. For $x^3 - 22x^2 + 80x - 67$: $p + q + r = 22$ and $pq + qr + rp = 80$. Then

$$p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + qr + rp) = 484 - 160 = 324,$$

and the answer is $324 - 80 = 244$.

(Numerically the roots are about 17.69, 3.08, 1.23, and computing A , B , C from them gives the same sum, 244.)

Another way. (For readers who know derivatives.) Step 2 says $\frac{1}{A} = (p - q)(p - r)$, which is exactly the derivative of $f(x) = (x - p)(x - q)(x - r)$ evaluated at p . So the sum is $f'(p) + f'(q) + f'(r)$ with $f'(x) = 3x^2 - 44x + 80$:

$$3(p^2 + q^2 + r^2) - 44(p + q + r) + 3 \cdot 80 = 3 \cdot 324 - 44 \cdot 22 + 240 = 972 - 968 + 240 = 244.$$

Answer: (B) 244

25 Number Theory

Source: 2014 AMC 10A #25

The number 5^{867} is between 2^{2013} and 2^{2014} . How many pairs of integers (m, n) are there such that $1 \leq m \leq 2012$ and

$$5^n < 2^m < 2^{m+2} < 5^{n+1}?$$

- (A) 278 (B) 279 (C) 280 (D) 281 (E) 282

Key idea. The condition says three consecutive powers of 2 — $2^m, 2^{m+1}, 2^{m+2}$ — fit strictly between two consecutive powers of 5. Since $4 < 5 < 8$, a gap from 5^n to 5^{n+1} can hold only two or three powers of 2. The odd-looking fact $2^{2013} < 5^{867} < 2^{2014}$ tells you the *total* number of powers of 2 in all the gaps — so count the gaps two ways.

Step 1: each gap holds two or three. Call the open interval $(5^n, 5^{n+1})$ “gap n ”, for $n = 0, 1, \dots, 866$. Let 2^k be the smallest power of 2 above 5^n , so $2^{k-1} \leq 5^n$. Then $2^{k+1} = 4 \cdot 2^{k-1} \leq 4 \cdot 5^n < 5^{n+1}$, so 2^k and 2^{k+1} both lie in gap n . And $2^{k+3} = 8 \cdot 2^k > 8 \cdot 5^n > 5^{n+1}$, so 2^{k+3} does not. Hence every gap contains exactly 2 or 3 powers of 2.

Step 2: pairs $\leftrightarrow$ gaps with three. A pair (m, n) works exactly when $2^m, 2^{m+1}, 2^{m+2}$ all lie in gap n . That happens only in a gap with three powers, and then 2^m must be the smallest of them. So each three-power gap gives exactly one pair, and two-power gaps give none.

Check the ranges. For any valid pair, $5^{n+1} > 2^{m+2} > 1$ gives $n \geq 0$, and $5^n < 2^m \leq 2^{2012} < 5^{867}$ gives $n \leq 866$. Conversely, in a three-power gap $n \leq 866$ the smallest power 2^m has $2^m > 5^n \geq 1$, so $m \geq 1$, and $2^{m+2} < 5^{867} < 2^{2014}$, so $m \leq 2011$. So the pairs correspond exactly to the three-power gaps among gaps 0 to 866.

Step 3: count two ways. Let d gaps hold two powers and t gaps hold three. There are 867 gaps (from $n = 0$ to $n = 866$), so $d + t = 867$. The powers $2^1, 2^2, \dots, 2^{2013}$ all lie strictly between 1 and 5^{867} , and none of them equals a power of 5, so each one lies in exactly one gap. So $2d + 3t = 2013$.

Step 4: solve. Subtracting twice the first equation: $t = 2013 - 2 \cdot 867 = 2013 - 1734 = 279$. (Then $d = 588$.) A direct computer count of all pairs also gives 279.

Watch out. Off-by-one errors land on neighbouring choices. Starting the gaps at $n = 1$ (only 866 gaps) while still counting all 2013 powers gives $2013 - 1732 = 281$, choice (D); using 2012 powers (from $m \leq 2012$) with 867 gaps gives 278, choice (A). Gap 0 is $(1, 5)$ and contains 2 and 4, and the powers inside are 2^1 through 2^{2013} .

Answer: (B) 279