
AMC 10

SET 3

PRACTICE EXAMINATION

SOLUTIONS

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All 25 problems are taken from past AMC 10 contests (2000–2019) and appear in their original wording, rearranged into a new paper. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	B	2018 AMC 10A #1	Algebra
2	B	2016 AMC 10B #2	Algebra
3	D	2003 AMC 10A #3	Geometry
4	B	2005 AMC 10A #4	Geometry
5	C	2014 AMC 10A #5	Algebra
6	D	2009 AMC 10B #6	Number Theory
7	B	2011 AMC 10A #7	Algebra
8	D	2010 AMC 10A #8	Algebra
9	D	2002 AMC 10B #9	Counting & Probability
10	B	2009 AMC 10A #10	Geometry
11	C	2014 AMC 10B #11	Algebra
12	B	2013 AMC 10B #12	Counting & Probability
13	C	2005 AMC 10B #13	Number Theory
14	B	2012 AMC 10A #14	Counting & Probability
15	A	2018 AMC 10B #15	Geometry
16	B	2000 AMC 10 #16	Geometry
17	B	2017 AMC 10B #17	Counting & Probability
18	D	2017 AMC 10A #18	Counting & Probability
19	C	2006 AMC 10A #19	Algebra
20	D	2010 AMC 10B #20	Geometry
21	B	2011 AMC 10B #21	Algebra
22	A	2015 AMC 10A #22	Counting & Probability
23	C	2019 AMC 10A #23	Algebra
24	D	2008 AMC 10A #24	Number Theory
25	C	2002 AMC 10A #25	Geometry

1 Algebra

Source: 2018 AMC 10A #1

What is the value of

$$\left(\left((2+1)^{-1} + 1 \right)^{-1} + 1 \right)^{-1} + 1?$$

- (A) $\frac{5}{8}$ (B) $\frac{11}{7}$ (C) $\frac{8}{5}$ (D) $\frac{18}{11}$ (E) $\frac{15}{8}$

Key idea. Work from the inside out. A power of -1 just means “take the reciprocal”, so each layer is “flip, then add 1”.

Step 1. Innermost: $(2+1)^{-1} = 3^{-1} = \frac{1}{3}$. Adding 1 gives $\frac{1}{3} + 1 = \frac{4}{3}$.

Step 2. Flip: $\left(\frac{4}{3}\right)^{-1} = \frac{3}{4}$. Adding 1 gives $\frac{3}{4} + 1 = \frac{7}{4}$.

Step 3. Flip: $\left(\frac{7}{4}\right)^{-1} = \frac{4}{7}$. Adding the final 1 gives $\frac{4}{7} + 1 = \frac{11}{7}$.

Watch out. There are three flips but four “+1”s. Stopping after the last flip gives $\frac{4}{7}$, which is not a choice — check that you have used the +1 outside the big parentheses.

Answer: (B) $\frac{11}{7}$

2 Algebra

Source: 2016 AMC 10B #2

If $n \heartsuit m = n^3 m^2$, what is $\frac{2 \heartsuit 4}{4 \heartsuit 2}$?

- (A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 4

Key idea. Just apply the rule: the *first* number is cubed and the *second* is squared. Then simplify the fraction.

Numerator. $2 \heartsuit 4 = 2^3 \cdot 4^2 = 8 \cdot 16 = 128$.

Denominator. $4 \heartsuit 2 = 4^3 \cdot 2^2 = 64 \cdot 4 = 256$.

So $\frac{2 \heartsuit 4}{4 \heartsuit 2} = \frac{128}{256} = \frac{1}{2}$.

Another way. In general $\frac{a \heartsuit b}{b \heartsuit a} = \frac{a^3 b^2}{b^3 a^2} = \frac{a}{b}$, so the answer is $\frac{2}{4} = \frac{1}{2}$ with no big numbers at all.

Watch out. The operation is not symmetric. Cubing the second number instead of the first turns the fraction upside down and gives 2, choice (D).

Answer: (B) $\frac{1}{2}$

3 Geometry

Source: 2003 AMC 10A #3

A solid box is 15 cm by 10 cm by 8 cm. A new solid is formed by removing a cube 3 cm on a side from each corner of this box. What percent of the original volume is removed?

- (A) 4.5 (B) 9 (C) 12 (D) 18 (E) 24

Key idea. Find the volume removed and the original volume, then turn their ratio into a percent. The key count: a box has *eight* corners.

Step 1. Each cube has volume $3^3 = 27 \text{ cm}^3$. A box has 8 corners (four on top, four on the bottom), so the removed volume is $8 \cdot 27 = 216 \text{ cm}^3$.

Step 2. The original box has volume $15 \cdot 10 \cdot 8 = 1200 \text{ cm}^3$.

Step 3. The fraction removed is $\frac{216}{1200} = \frac{18}{100}$, which is 18%.

Watch out. Counting only the four corners you see on top gives $4 \cdot 27 = 108 \text{ cm}^3$, which is 9% — choice (B). A rectangular box has eight corners.

Answer: (D) 18

4 Geometry

Source: 2005 AMC 10A #4

A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle?

- (A) $\frac{1}{4}x^2$ (B) $\frac{2}{5}x^2$ (C) $\frac{1}{2}x^2$ (D) x^2 (E) $\frac{3}{2}x^2$

Key idea. The diagonal is the hypotenuse of a right triangle whose legs are the width and the length, so use the Pythagorean Theorem.

Step 1. Let the width be w ; then the length is $2w$. The diagonal splits the rectangle into two right triangles with legs w and $2w$, so

$$x^2 = w^2 + (2w)^2 = 5w^2, \quad \text{so} \quad w^2 = \frac{x^2}{5}.$$

Step 2. The area is $w \cdot 2w = 2w^2 = 2 \cdot \frac{x^2}{5} = \frac{2}{5}x^2$.

Notice that we never needed w itself — only w^2 , which the Pythagorean Theorem hands us directly.

Watch out. The area is $2w^2$, not w^2 . Stopping at $w^2 = \frac{1}{5}x^2$ gives a value that is not among the choices.

Answer: (B) $\frac{2}{5}x^2$

5 Algebra

Source: 2014 AMC 10A #5

On an algebra quiz, 10% of the students scored 70 points, 35% scored 80 points, 30% scored 90 points, and the rest scored 100 points. What is the difference between the mean and median score of the students' scores on this quiz?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Key idea. Percentages behave like weights. The mean is a weighted average; the median is the score of the student in the middle when everyone is lined up.

Step 1. The rest is $100\% - 10\% - 35\% - 30\% = 25\%$ scoring 100.

Step 2 (median). Line the students up from lowest to highest. The first $10\% + 35\% = 45\%$ scored below 90, and the first $45\% + 30\% = 75\%$ scored 90 or less. The middle of the line (50%) falls between 45% and 75%, so the median is 90.

Step 3 (mean). Weight each score by its share:

$$0.10 \cdot 70 + 0.35 \cdot 80 + 0.30 \cdot 90 + 0.25 \cdot 100 = 7 + 28 + 27 + 25 = 87.$$

The difference is $90 - 87 = 3$.

Watch out. Averaging the four score values 70, 80, 90, 100 without weights gives 85 and a difference of 5, choice (E). The mean must use the percentages.

Answer: (C) 3

6 Number Theory

Source: 2009 AMC 10B #6

Kiana has two older twin brothers. The product of their three ages is 128. What is the sum of their three ages?

- (A) 10 (B) 12 (C) 16 (D) 18 (E) 24

Key idea. $128 = 2^7$ has no odd factor other than 1, so every age is a power of 2. Then use the two facts given: the twins share an age, and they are *older* than Kiana.

Step 1. Let each twin be t years old and Kiana be k . Then $t \cdot t \cdot k = t^2 k = 128$, and every age divides 128, so every age is a power of 2.

Step 2. Try each possible twin age with t^2 dividing 128:

- $t = 1$: $k = 128$; $t = 2$: $k = 32$; $t = 4$: $k = 8$;
- $t = 8$: $k = \frac{128}{64} = 2$.

($t = 16$ would need $t^2 = 256 > 128$.)

Step 3. Only $t = 8$, $k = 2$ has the twins older than Kiana. The sum is $8 + 8 + 2 = 18$.

Watch out. Ignoring the word “older” lets you pick $t = 4$, $k = 8$, with sum $4 + 4 + 8 = 16$ — choice (C). There Kiana would be older than her brothers.

Answer: (D) 18**7** Algebra

Source: 2011 AMC 10A #7

Which of the following equations does not have a solution?

- (A) $(x + 7)^2 = 0$ (B) $|-3x| + 5 = 0$ (C) $\sqrt{-x} - 2 = 0$
(D) $\sqrt{x} - 8 = 0$ (E) $|-3x| - 4 = 0$

Key idea. An absolute value can never be negative. Isolate the absolute value or the square root in each equation and see whether what is left is allowed.

(A) $(x + 7)^2 = 0$ means $x + 7 = 0$, so $x = -7$ works.

(B) $|-3x| + 5 = 0$ means $|-3x| = -5$. An absolute value is always 0 or more, so it can never equal -5 . *No solution.*

(C) $\sqrt{-x} = 2$, so $-x = 4$ and $x = -4$. Check: $\sqrt{-(-4)} = \sqrt{4} = 2$. It works.

(D) $\sqrt{x} = 8$, so $x = 64$.

(E) $|-3x| = 4$, so $-3x = \pm 4$ and $x = \pm \frac{4}{3}$.

Only (B) has no solution.

Watch out. $\sqrt{-x}$ looks like “the square root of a negative”, which tempts you to pick (C). But when x is negative, $-x$ is positive: $x = -4$ is a perfectly good solution.

Answer: (B) $|-3x| + 5 = 0$

8 Algebra

Source: 2010 AMC 10A #8

Tony works 2 hours a day and is paid \$0.50 per hour for each full year of his age. During a six month period Tony worked 50 days and earned \$630. How old was Tony at the end of the six month period?

- (A) 9 (B) 11 (C) 12 (D) 13 (E) 14

Key idea. Find his *average* pay per hour, which tells you his average age. If that average is not a whole number, he must have had a birthday during the period.

Step 1. He worked $2 \cdot 50 = 100$ hours, so he averaged $\$630 \div 100 = \6.30 per hour.

Step 2. He earns \$0.50 per hour per year of age, so his average age while working was $6.30 \div 0.50 = 12.6$ years.

Step 3. In six months his age changes at most once. An average of 12.6 means he worked some days at age 12 and some at age 13. Check: if he worked d days at 12 and $50 - d$ at 13, then his pay is \$1 per day per year of age, so $12d + 13(50 - d) = 630$, giving $650 - d = 630$ and $d = 20$. So 20 days at 12, then 30 days at 13.

He was 13 at the end of the period.

Watch out. Truncating 12.6 to 12 gives his age at the *start* of the period — choice (C). The question asks about the end.

Answer: (D) 13

9 Counting & Probability

Source: 2002 AMC 10B #9

Using the letters $A, M, O, S,$ and U , we can form 120 five-letter “words.” If these “words” are arranged in alphabetical order, then the “word” $USAMO$ occupies which position?

- (A) 112 (B) 113 (C) 114 (D) 115 (E) 116

Key idea. Count how many words come *before* $USAMO$, in blocks sorted by first letter, then by second letter.

Step 1. The letters in order are A, M, O, S, U . With a fixed first letter, the other four letters can be arranged in $4! = 24$ ways.

Step 2. Words starting with $A, M, O,$ or S all come before $USAMO$: that is $4 \cdot 24 = 96$ words.

Step 3. Among words starting with U , the second letter can be A, M, O, S , each followed by $3! = 6$ arrangements. Words starting UA, UM, UO come first: $3 \cdot 6 = 18$ more words.

Step 4. Among words starting US , the remaining letters A, M, O in alphabetical order give $USAMO$, the very first one.

So $96 + 18 = 114$ words come before $USAMO$, and it sits in position 115.

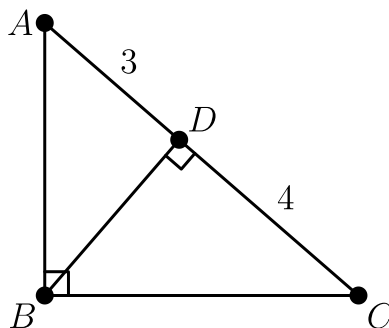
Watch out. 114 is the number of words *before* $USAMO$, not its position. Forgetting to add 1 gives choice (C).

Answer: (D) 115

10 Geometry

Source: 2009 AMC 10A #10

Triangle ABC has a right angle at B . Point D is the foot of the altitude from B , $AD = 3$, and $DC = 4$. What is the area of $\triangle ABC$?



- (A) $4\sqrt{3}$ (B) $7\sqrt{3}$ (C) 21 (D) $14\sqrt{3}$ (E) 42

Key idea. We know the base $AC = 7$; we need the height BD . The altitude to the hypotenuse cuts a right triangle into two smaller triangles similar to it, and that gives $BD^2 = AD \cdot DC$.

Step 1. In right triangle ABD , $\angle A + \angle ABD = 90^\circ$. Also $\angle ABD + \angle DBC = 90^\circ$ (the right angle at B). So $\angle A = \angle DBC$, and right triangles ADB and BDC are similar.

Step 2. Matching sides, $\frac{AD}{BD} = \frac{BD}{DC}$, so

$$BD^2 = AD \cdot DC = 3 \cdot 4 = 12, \quad BD = \sqrt{12} = 2\sqrt{3}.$$

Step 3. Take $AC = 3 + 4 = 7$ as the base and BD as the height:

$$\text{Area} = \frac{1}{2} \cdot 7 \cdot 2\sqrt{3} = 7\sqrt{3}.$$

Watch out. Using $BD^2 = 12$ as if it were BD gives $\frac{1}{2} \cdot 7 \cdot 12 = 42$ — choice (E). Forgetting the $\frac{1}{2}$ gives $14\sqrt{3}$ — choice (D).

Answer: (B) $7\sqrt{3}$

11 Algebra

Source: 2014 AMC 10B #11

For the consumer, a single discount of $n\%$ is more advantageous than any of the following discounts:

- (1) Two successive 15% discounts.
- (2) Three successive 10% discounts.
- (3) A 25% discount followed by a 5% discount.

What is the smallest possible positive integer value of n ?

- (A) 27 (B) 28 (C) 29 (D) 31 (E) 33

Key idea. Successive discounts *multiply*, they do not add. Turn each plan into “the fraction of the price you still pay”, then find the single discount that beats all three. The words “successive” and “followed by” are the signal to multiply.

Step 1. A 15% discount means you pay 0.85 of the price. Two of them in a row leave

$$0.85 \times 0.85 = 0.7225,$$

an effective discount of 27.75%.

Step 2. Three successive 10% discounts leave

$$0.9 \times 0.9 \times 0.9 = 0.729,$$

an effective discount of 27.1%.

Step 3. A 25% discount followed by a 5% discount leaves

$$0.75 \times 0.95 = 0.7125,$$

an effective discount of 28.75%.

Step 4. “More advantageous” means a bigger discount, so n must beat the best of the three: $n > 28.75$. The smallest integer that works is $n = 29$. Check: $n = 28$ fails, since $28\% < 28.75\%$, while 29% beats 27.75%, 27.1% and 28.75%.

Watch out. Adding the discounts gives $15 + 15 = 30$, $10 + 10 + 10 = 30$ and $25 + 5 = 30$, so you would need $n > 30$ and pick 31 — choice (D). The second discount is taken off an already reduced price, so it is worth less than its face value.

Watch out. Beating only the first two plans needs just $n > 27.75$, giving 28 — choice (B). The single discount must beat *every* plan, including the best one.

Answer: (C) 29

12 Counting & Probability

Source: 2013 AMC 10B #12

Let S be the set of sides and diagonals of a regular pentagon. A pair of elements of S are selected at random without replacement. What is the probability that the two chosen segments have the same length?

- (A) $\frac{2}{5}$ (B) $\frac{4}{9}$ (C) $\frac{1}{2}$ (D) $\frac{5}{9}$ (E) $\frac{4}{5}$

Key idea. In a regular pentagon there are only two lengths: all 5 sides are equal, and all 5 diagonals are equal (and longer). So the question is simply “do both picks come from the same group of 5?”. Whenever you choose a pair, it is easiest to fix the first pick and ask about the second.

Step 1. A pentagon has 5 sides and $\binom{5}{2} - 5 = 10 - 5 = 5$ diagonals, so S has 10 segments: 5 of one length and 5 of another.

Step 2. Pick the first segment; whatever it is, it belongs to a group of 5 with the same length. That leaves 9 segments, of which 4 are in the same group.

Step 3. So the probability that the second matches the first is $\frac{4}{9}$.

Another way. Count pairs directly. There are $\binom{10}{2} = 45$ pairs in all. Matching pairs are two sides, $\binom{5}{2} = 10$ ways, or two diagonals, 10 ways: 20 in all. The probability is $\frac{20}{45} = \frac{4}{9}$.

Watch out. Thinking “half the segments match, so the answer is $\frac{5}{10}$ ” gives $\frac{1}{2}$ — choice (C). Once the first segment is taken, only 4 of its group are left among the 9 remaining.

Answer: (B) $\frac{4}{9}$
13 Number Theory

Source: 2005 AMC 10B #13

How many numbers between 1 and 2005 are integer multiples of 3 or 4 but not 12?

- (A) 501 (B) 668 (C) 835 (D) 1002 (E) 1169

Key idea. Count multiples of 3 and of 4 with the floor function. The numbers that are multiples of *both* are exactly the multiples of 12 — the very numbers we must throw out. So remove them from each list.

Step 1. Count each type up to 2005:

- multiples of 3: $\left\lfloor \frac{2005}{3} \right\rfloor = 668$ (since $3 \cdot 668 = 2004$);
- multiples of 4: $\left\lfloor \frac{2005}{4} \right\rfloor = 501$ (since $4 \cdot 501 = 2004$);
- multiples of 12: $\left\lfloor \frac{2005}{12} \right\rfloor = 167$ (since $12 \cdot 167 = 2004$).

Step 2. A number is a multiple of both 3 and 4 exactly when it is a multiple of 12. So the multiples of 3 that are *not* multiples of 12 number $668 - 167 = 501$, and the multiples of 4 that are not multiples of 12 number $501 - 167 = 334$.

Step 3. These two groups do not overlap (anything in both would be a multiple of 12), so the total is $501 + 334 = 835$.

Neither endpoint matters: 1 and 2005 are multiples of neither 3 nor 4.

Watch out. Ordinary inclusion–exclusion, $668 + 501 - 167 = 1002$, counts the multiples of 3 *or* 4 — but that still includes each multiple of 12 once. That is choice (D); you must subtract the 167 a second time.

Answer: (C) 835

14 Counting & Probability

Source: 2012 AMC 10A #14

Chubby makes nonstandard checkerboards that have 31 squares on each side. The checkerboards have a black square in every corner and alternate red and black squares along every row and column. How many black squares are there on such a checkerboard?

- (A) 480 (B) 481 (C) 482 (D) 483 (E) 484

Key idea. 31 is odd, so a row that starts and ends with black has one more black than red. Count row by row — the corners tell you which rows start with black.

Step 1. Number the rows 1 to 31. Row 1 starts with a black corner and alternates, so its squares are black in positions 1, 3, 5, ..., 31: that is 16 black and 15 red.

Step 2. Colors also alternate down each column, so row 2 starts with red and has 15 black squares, row 3 starts with black again, and so on. The odd-numbered rows 1, 3, ..., 31 — there are 16 of them — each have 16 black squares. The even-numbered rows 2, 4, ..., 30 — there are 15 of them — each have 15.

Step 3. The total is $16 \cdot 16 + 15 \cdot 15 = 256 + 225 = 481$.

Check: the board has $31^2 = 961$ squares, and 481 black plus 480 red makes 961.

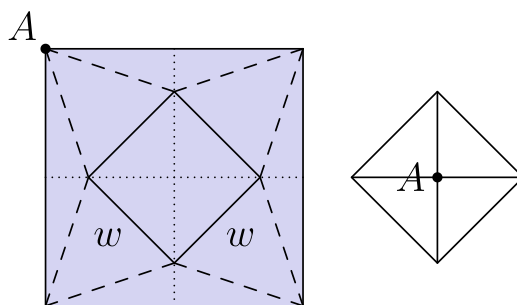
Watch out. Splitting the board evenly, $\frac{961}{2} = 480.5$, and rounding down gives 480 — choice (A). That is the number of *red* squares; black is the color that gets the extra square.

Answer: (B) 481

15 Geometry

Source: 2018 AMC 10B #15

A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h . What is the area of the sheet of wrapping paper?



- (A) $2(w + h)^2$ (B) $\frac{(w + h)^2}{2}$ (C) $2w^2 + 4wh$ (D) $2w^2$ (E) w^2h

Key idea. Follow one corner of the paper as it is folded. The paper does not stretch, so the distance from the center of the sheet to a corner equals the length of the path that corner travels over the box to reach A . A question about folding is almost always a question about lengths that stay the same.

Step 1 (where the corner starts). Let the sheet have side s . Its center is the center of the box's base, and each corner of the sheet is half a diagonal away: $\frac{s\sqrt{2}}{2} = \frac{s}{\sqrt{2}}$.

Step 2 (which way it travels). The vertices of the base lie on the midlines, so the base is a square turned 45° , and each of its sides is perpendicular to one diagonal of the sheet. So the line from the center to a sheet corner runs straight out and crosses one base edge at a right angle, at that edge's midpoint.

Step 3 (the length of the path). Along that line the corner's paper covers:

- from the center to the base edge: $\frac{w}{2}$ (half the side of the base);
- up the side of the box: h ;
- across the top from the edge to the center A : $\frac{w}{2}$.

In total $\frac{w}{2} + h + \frac{w}{2} = w + h$.

Step 4. Setting the two lengths equal, $\frac{s}{\sqrt{2}} = w + h$, so $s = \sqrt{2}(w + h)$ and the area is

$$s^2 = 2(w + h)^2.$$

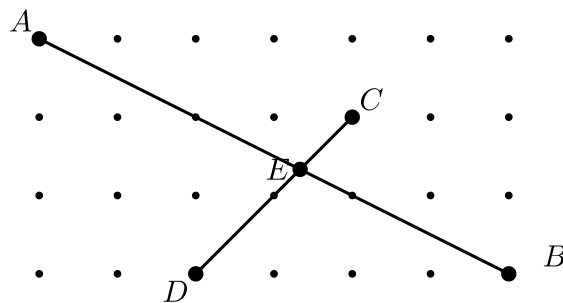
Watch out. The paper only needs to cover the box, so it is tempting to answer with the box's surface area, $2w^2 + 4wh$ — choice (C). But the folded flaps overlap: $2(w + h)^2 = 2w^2 + 4wh + 2h^2$, which is $2h^2$ more.

Answer: (A) $2(w + h)^2$

16 Geometry

Source: 2000 AMC 10 #16

The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E . Find the length of segment AE .



- (A) $\frac{4\sqrt{5}}{3}$ (B) $\frac{5\sqrt{5}}{3}$ (C) $\frac{12\sqrt{5}}{7}$ (D) $2\sqrt{5}$ (E) $\frac{5\sqrt{65}}{9}$

Key idea. Every labelled point except E sits on the lattice, and that is the hint: give the points coordinates. The two segments become two lines with exact equations, and E is simply where they cross.

Step 1. Put the bottom-left lattice point at the origin. The grid is 7 points wide ($x = 0$ to 6) and 4 points tall ($y = 0$ to 3), so $7 \cdot 4 = 28$ points. Reading off the diagram,

$$A = (0, 3), \quad B = (6, 0), \quad C = (4, 2), \quad D = (2, 0).$$

Step 2. Line AB drops 3 while moving 6 to the right, so its slope is $-\frac{1}{2}$ and its equation is $y = 3 - \frac{x}{2}$. Line CD rises 2 while moving 2 to the right, so its slope is 1 and its equation is $y = x - 2$.

Step 3. At E both equations hold: $3 - \frac{x}{2} = x - 2$, so $5 = \frac{3x}{2}$ and $x = \frac{10}{3}$. Then $y = \frac{10}{3} - 2 = \frac{4}{3}$, so $E = (\frac{10}{3}, \frac{4}{3})$.

Step 4. From $A = (0, 3)$ to E the horizontal change is $\frac{10}{3}$ and the vertical change is $3 - \frac{4}{3} = \frac{5}{3}$. By the Pythagorean theorem,

$$AE = \sqrt{\frac{100}{9} + \frac{25}{9}} = \sqrt{\frac{125}{9}} = \frac{5\sqrt{5}}{3}.$$

Another way. Extend $\overline{DC}$ past C . It rises 1 for each step right, so it reaches the lattice point $F = (5, 3)$ in the top row, the same row as A . Now $\overline{AF}$ and $\overline{DB}$ are both horizontal, hence parallel, so

$\triangle AEF \sim \triangle BED$. Since $AF = 5$ and $BD = 6 - 2 = 4$, we get $AE : EB = 5 : 4$. The whole segment is $AB = \sqrt{6^2 + 3^2} = 3\sqrt{5}$, so $AE = \frac{5}{9} \cdot 3\sqrt{5} = \frac{5\sqrt{5}}{3}$.

Watch out. In the similar-triangle method, AE sits next to the longer parallel side $AF = 5$, so it gets 5 of the 9 parts. Giving it 4 parts instead yields $\frac{4}{9} \cdot 3\sqrt{5} = \frac{4\sqrt{5}}{3}$, choice (A).

Answer: (B) $\frac{5\sqrt{5}}{3}$

17 Counting & Probability

Source: 2017 AMC 10B #17

Call a positive integer *monotonous* if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

- (A) 1024 (B) 1524 (C) 1533 (D) 1536 (E) 2048

Key idea. Counting numbers digit by digit is hopeless here, but notice that a strictly increasing number is completely determined by *which* digits it uses — the order is forced. So count sets of digits instead of numbers, then fix the overlap between the two families.

Step 1: increasing. Choose any nonempty set of digits and write them in increasing order. The digit 0 cannot be used: it would be the smallest, so it would come first, and a positive integer cannot start with 0. So the increasing numbers match the nonempty subsets of $\{1, 2, \dots, 9\}$:

$$2^9 - 1 = 511.$$

Step 2: decreasing. Now write a chosen set of digits in decreasing order. Here 0 is allowed, because it would come *last* (as in 987620). So use subsets of $\{0, 1, \dots, 9\}$, of which there are $2^{10} = 1024$. Throw out the empty set and the set $\{0\}$ (which would give the number 0, not a positive integer):

$$1024 - 2 = 1022.$$

Step 3: overlap. A one-digit number $1, 2, \dots, 9$ is a set with a single element, so it was produced in both Step 1 and Step 2. Every number with two or more digits is either increasing or decreasing, never both. So exactly 9 numbers were counted twice:

$$511 + 1022 - 9 = 1524.$$

Watch out. Two easy slips each land on a choice. Forgetting that 1 through 9 were counted in both families gives $511 + 1022 = 1533$, choice (C). Just adding the subset counts $2^9 + 2^{10} = 1536$ — keeping the empty set and $\{0\}$ and ignoring the double count — gives choice (D).

Answer: (B) 1524

18 Counting & Probability

Source: 2017 AMC 10A #18

Amelia has a coin that lands heads with probability $\frac{1}{3}$, and Blaine has a coin that lands on heads with probability $\frac{2}{5}$. Amelia and Blaine alternately toss their coins until someone gets a head; the first one to get a head wins. All coin tosses are independent. Amelia goes first. The probability that Amelia wins is $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $q - p$?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Key idea. The game could last forever, so you cannot list every case. But after one full round with no winner (Amelia tails, then Blaine tails) the game is *exactly* as it was at the start. That self-similarity lets you write one equation for Amelia's winning probability.

Step 1. Let x be the probability that Amelia wins. On her first toss she wins at once with probability $\frac{1}{3}$.

Step 2. Otherwise she tosses tails (probability $\frac{2}{3}$) and then Blaine also tosses tails (probability $1 - \frac{2}{5} = \frac{3}{5}$). Both happen with probability

$$\frac{2}{3} \cdot \frac{3}{5} = \frac{6}{15} = \frac{2}{5}.$$

Then it is Amelia's turn again with nothing changed, so from there she wins with probability x again. In every other case Blaine has won.

Step 3. Putting the two winning routes together,

$$x = \frac{1}{3} + \frac{2}{5}x \implies \frac{3}{5}x = \frac{1}{3} \implies x = \frac{5}{9}.$$

Since 5 and 9 are relatively prime, $p = 5$, $q = 9$, and $q - p = 4$.

Another way. Look only at the round that ends the game. In any single round, Amelia wins with probability $\frac{1}{3} = \frac{5}{15}$, Blaine wins with probability $\frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}$, and otherwise the round is replayed. Replays do not favor anyone, so the winning chances stay in the ratio 5 : 4, and Amelia wins with probability $\frac{5}{5+4} = \frac{5}{9}$.

Watch out. Check whose probability you found. Blaine's chance is $1 - \frac{5}{9} = \frac{4}{9}$, and reporting that one gives $q - p = 9 - 4 = 5$, choice (E).

Answer: (D) 4

19 Algebra

Source: 2006 AMC 10A #19

How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?

- (A) 0 (B) 1 (C) 59 (D) 89 (E) 178

Key idea. Three numbers in arithmetic progression are best written *symmetrically* about the middle one, as $n - d$, n , $n + d$: then their sum is just $3n$. The angle sum 180° pins down n at once, and only d is left to count. “Non-similar” tells you that a triangle is determined by its three angles.

Step 1. List the angles from smallest to largest as $n - d$, n , $n + d$ with $d \geq 0$. Their sum is

$$(n - d) + n + (n + d) = 3n = 180,$$

so the middle angle is always $n = 60^\circ$.

Step 2. Now find the allowed d . The angles must be *distinct*, so $d \neq 0$, that is $d \geq 1$. The smallest angle must be *positive*: $60 - d > 0$, so $d \leq 59$. Every whole number d from 1 to 59 works, giving the angles

$$59^\circ, 60^\circ, 61^\circ, \quad 58^\circ, 60^\circ, 62^\circ, \quad \dots, \quad 1^\circ, 60^\circ, 119^\circ.$$

Step 3. Two triangles are similar exactly when they have the same three angles. Different values of d give different smallest angles, so these 59 triangles are pairwise non-similar, and Step 1 shows there are no others.

Watch out. Do not count d and $-d$ separately: $60 - d, 60, 60 + d$ and $60 + d, 60, 60 - d$ are the same three angles, so the same triangle up to similarity. Counting both would give 118, which is not a choice. Likewise $d = 0$ (the equilateral triangle) is excluded by the word “distinct”.

Answer: (C) 59

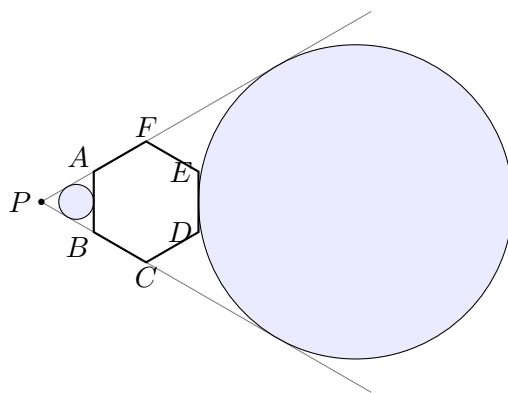
20 Geometry

Source: 2010 AMC 10B #20

Two circles lie outside regular hexagon $ABCDEF$. The first is tangent to $\overline{AB}$, and the second is tangent to $\overline{DE}$. Both are tangent to lines BC and FA . What is the ratio of the area of the second circle to that of the first circle?

- (A) 18 (B) 27 (C) 36 (D) 81 (E) 108

Key idea. The phrase “both are tangent to lines BC and FA ” is the signal: extend those two lines until they meet at a point P . Both circles then sit inside the same angle at P , and for a circle in an angle the radius is a fixed fraction of the distance from P to the center. So everything reduces to measuring distances from P .



Step 1. Take the side of the hexagon to be 1. Each interior angle of a regular hexagon is 120° , so lines BC and FA each make a $180^\circ - 120^\circ = 60^\circ$ angle with $\overline{AB}$. Hence they meet at a point P beyond $\overline{AB}$, and $\triangle PAB$ is equilateral with side 1. Its angle at P is 60° and its height from P is $\frac{\sqrt{3}}{2}$.

Step 2. A circle tangent to both sides of this 60° angle has its center on the bisector, the line through P and the center of the hexagon. If the center is at distance d from P , the radius is $d \sin 30^\circ = \frac{d}{2}$. So for each circle, *distance from P to the center* $= 2r$.

Step 3. The first circle touches the three sides of $\triangle PAB$ and stays outside the hexagon, so it is the incircle of $\triangle PAB$. Its center is $\frac{2}{3}$ of the way down the height, at distance $\frac{2}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}$ from P , so $r_1 = \frac{\sqrt{3}}{6}$.

Step 4. The second circle touches line DE from the far side. The distance from P to line DE is the triangle's height plus the hexagon's height (twice the distance from its center to a side): $\frac{\sqrt{3}}{2} + \sqrt{3} = \frac{3\sqrt{3}}{2}$. Its center is a further r_2 beyond DE , so by Step 2

$$2r_2 = \frac{3\sqrt{3}}{2} + r_2 \implies r_2 = \frac{3\sqrt{3}}{2}.$$

Step 5. The radii are in the ratio $\frac{r_2}{r_1} = \frac{3\sqrt{3}/2}{\sqrt{3}/6} = 9$, so the areas are in the ratio $9^2 = 81$.

Watch out. The three lines BC , FA , DE form an equilateral triangle of side 3, and its *incircle* (radius $\frac{\sqrt{3}}{2}$, ratio of areas 9) is tempting. But that circle is just the hexagon's own inscribed circle, which lies *inside* the hexagon; the second circle must lie outside the hexagon, beyond $\overline{DE}$. Also remember to square the ratio of radii.

Answer: (D) 81

21 Algebra

Source: 2011 AMC 10B #21

Brian writes down four integers $w > x > y > z$ whose sum is 44. The pairwise positive differences of these numbers are 1, 3, 4, 5, 6, and 9. What is the sum of the possible values for w ?

- (A) 16 (B) 31 (C) 48 (D) 62 (E) 93

Key idea. Picture the four numbers as points on a number line. The largest difference must be the distance between the two ends, and each middle point splits that distance into two gaps that *add up* to it. Looking for pairs that sum to 9 cuts the possibilities down to almost nothing.

Step 1: the ends. The biggest of the six differences is the distance from the largest number to the smallest, so $w - z = 9$.

Step 2: pairs that add to 9. For the middle number x , the two gaps $w - x$ and $x - z$ add up to $w - z = 9$. The same is true for y : $(w - y) + (y - z) = 9$. These four gaps, together with $x - y$, are the five remaining differences 1, 3, 4, 5, 6, each used once. The value 1 cannot be in a pair adding to 9 (it would need an 8). So the two pairs are $\{3, 6\}$ and $\{4, 5\}$, and the leftover difference is $x - y = 1$.

Step 3: the cases. The pair $\{w - x, x - z\}$ is one of $\{3, 6\}$, $\{4, 5\}$, and $\{w - y, y - z\}$ is the other. We need $w - y = (w - x) + 1$:

- $w - x = 3$, $w - y = 4$: the numbers are w , $w - 3$, $w - 4$, $w - 9$.
- $w - x = 5$, $w - y = 6$: the numbers are w , $w - 5$, $w - 6$, $w - 9$.
- $w - x = 4$ would need $w - y = 5$, but 5 is already $x - z$; $w - x = 6$ would need $w - y = 7$. Both fail.

Step 4: use the sum. In the first case $4w - 16 = 44$, so $w = 15$: the numbers 15, 12, 11, 6 have differences 3, 4, 9, 1, 6, 5. In the second case $4w - 20 = 44$, so $w = 16$: the numbers 16, 11, 10, 7 have differences 5, 6, 9, 1, 4, 3. Both work, so the answer is $15 + 16 = 31$.

Watch out. The question asks for the *sum of the possible values*, a strong hint that there is more than one. Stopping after finding only 16, 11, 10, 7 gives 16, choice (A). (Finding only 15, 12, 11, 6 gives 15, which is not even a choice — a sign that something is missing.)

Answer: (B) 31

22 Counting & Probability

Source: 2015 AMC 10A #22

Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

- (A) $\frac{47}{256}$ (B) $\frac{3}{16}$ (C) $\frac{49}{256}$ (D) $\frac{25}{128}$ (E) $\frac{51}{256}$

Key idea. All $2^8 = 256$ outcomes are equally likely, so we just count the good ones. People in a row with “no two neighbors standing” are easy to count with a recursion; the circle is harder only because the ends touch. So break the circle: decide what one person does, and what is left is a row.

Step 1: people in a row. Let a_n be the number of stand/sit patterns for n people in a row with no two neighbors standing. Look at the last person. If they sit, the first $n - 1$ people form any good pattern: a_{n-1} ways. If they stand, the person before them must sit, and the first $n - 2$ form any good pattern: a_{n-2} ways. So $a_n = a_{n-1} + a_{n-2}$.

Step 2: the numbers. One person can sit or stand, so $a_1 = 2$. Two people have 3 good patterns (not both standing), so $a_2 = 3$. Then

$$a_3 = 5, \quad a_4 = 8, \quad a_5 = 13, \quad a_6 = 21, \quad a_7 = 34.$$

Step 3: break the circle. Number the people 1 to 8 around the table, and look at person 1.

- Person 1 sits. Then persons 2, 3, ..., 8 remain. Persons 2 and 8 were neighbors only through person 1, so these seven form a row: $a_7 = 34$ patterns.
- Person 1 stands. Then both neighbors, persons 2 and 8, must sit, and persons 3, ..., 7 form a row of five: $a_5 = 13$ patterns.

So there are $34 + 13 = 47$ good outcomes, and the probability is $\frac{47}{256}$.

Another way. Count by the number of people standing. With 0 or 1 standing: $1 + 8 = 9$ ways. With 2: any pair except the 8 neighboring pairs, $\binom{8}{2} - 8 = 20$. With 3: pick one stander (8 ways); the other two come from the 5 seats not next to them, which form a row and must not be neighbors, $\binom{5}{2} - 4 = 6$ ways; each set was counted once for each of its 3 members, so $\frac{8 \cdot 6}{3} = 16$. With 4: only the two alternating patterns. Five or more standers would need at least five seated people between them, too many for 8 seats. Total $9 + 20 + 16 + 2 = 47$.

Watch out. Forgetting that the table is round (so persons 1 and 8 are neighbors) counts a row of eight: $a_8 = 55$ patterns and a probability of $\frac{55}{256}$, which is not among the choices.

Answer: (A) $\frac{47}{256}$

23 Algebra

Source: 2019 AMC 10A #23

Travis has to babysit the terrible Thompson triplets. Knowing that they love big numbers, Travis devises a counting game for them. First Tadd will say the number 1, then Todd must say the next two numbers (2 and 3), then Tucker must say the next three numbers (4, 5, 6), then Tadd must say the next four numbers (7, 8, 9, 10), and the process continues to rotate through the three children in order, each saying one more number than the previous child did, until the number 10,000 is reached. What is the 2019th number said by Tadd?

- (A) 5743 (B) 5885 (C) 5979 (D) 6001 (E) 6011

Key idea. Strip away the story: the integers are said in consecutive blocks of lengths $1, 2, 3, 4, \dots$, and Tadd owns every third block. Block sizes growing by one each time means triangular numbers — sums like $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ — so we can jump straight to the right block instead of simulating.

Step 1: Tadd's blocks. Call the blocks turns $1, 2, 3, \dots$; turn t contains t numbers. Tadd speaks on turns $1, 4, 7, \dots$, so his i -th turn is turn $3i - 2$ and contains $3i - 2$ numbers. (Check: his first turns are $\{1\}$, then $7-10$, then $22-28$.)

Step 2: which of Tadd's turns? After i of his turns, Tadd has said

$$1 + 4 + 7 + \dots + (3i - 2) = \frac{i(1 + (3i - 2))}{2} = \frac{i(3i - 1)}{2}$$

numbers. Since $\frac{3i^2}{2} \approx 2019$ gives $i \approx 37$, try nearby values: $i = 36$ gives $\frac{36 \cdot 107}{2} = 1926$, and $i = 37$ gives $\frac{37 \cdot 110}{2} = 2035$. So his 2019th number comes during his 37th turn, and it is number $2019 - 1926 = 93$ of that turn.

Step 3: where that turn starts. Tadd's 37th turn is turn $3 \cdot 37 - 2 = 109$ overall, which has $109 \geq 93$ numbers. Turns 1 through 108 used

$$1 + 2 + \dots + 108 = \frac{108 \cdot 109}{2} = 5886$$

numbers, so turn 109 begins with 5887. Its 93rd number is $5886 + 93 = 5979$, which is below 10,000, so the game is still running.

Watch out. Watch the off-by-one in Step 3: the block starts at 5887, so its 93rd number is $5887 + 92 = 5979$. Adding 93 to the starting number gives 5980, which is not a choice — a sign of a slip.

Answer: (C) 5979

24 Number Theory

Source: 2008 AMC 10A #24

Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

- (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Key idea. k is far too large to compute, but a units digit only depends on a little information. For k^2 we need the units digit of k . For 2^k we need something different: the units digits of powers of 2 repeat every 4, so we need the remainder when k is divided by 4.

Step 1: powers of 2. The units digits of $2^1, 2^2, 2^3, 2^4, 2^5, \dots$ are 2, 4, 8, 6, 2, $\dots$: each new digit is the units digit of twice the old one, so the pattern repeats with period 4. In particular, 2^n ends in 6 whenever n is a positive multiple of 4.

Step 2: the units digit of k . 2008^2 ends in the same digit as $8^2 = 64$, namely 4. Since 2008 is a multiple of 4, 2^{2008} ends in 6. So k ends in the units digit of $4 + 6 = 10$, which is 0, and therefore k^2 ends in 0.

Step 3: k divided by 4. $2008 = 4 \cdot 502$, so 2008^2 is a multiple of 4; and 2^{2008} is certainly a multiple of 4. Hence k is a positive multiple of 4, and by Step 1, 2^k ends in 6.

Step 4. The units digit of $k^2 + 2^k$ is the units digit of $0 + 6$, which is 6.

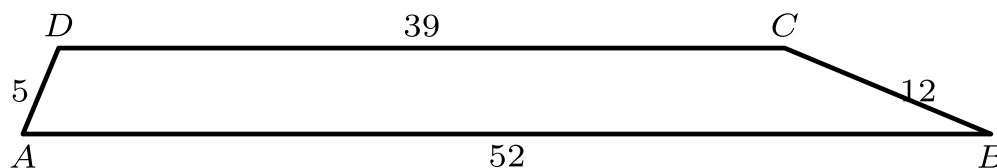
Watch out. Knowing that k ends in 0 is *not* enough for 2^k : $2^{10} = 1024$ ends in 4, but $2^{20} = 1048576$ ends in 6. Reasoning “ k ends in 0, like 10, so 2^k ends in 4” gives $0 + 4 = 4$, choice (C). The exponent’s remainder mod 4 is what matters.

Answer: (D) 6

25 Geometry

Source: 2002 AMC 10A #25

In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$. The area of $ABCD$ is



- (A) 182 (B) 195 (C) 210 (D) 234 (E) 260

Key idea. The area formula $\frac{1}{2}(AB + CD) \cdot h$ needs only the height h . A standard way to get it when all four sides are known is to slide one leg across until it touches the other: the trapezoid splits into a parallelogram and a triangle whose sides are the two legs and the *difference of the bases*. Here those are 5, 12 and $52 - 39 = 13$ — a Pythagorean triple, which is the giveaway.

Step 1: slide a leg. Through C draw the line parallel to $\overline{DA}$, meeting $\overline{AB}$ at E . Then $AECD$ is a parallelogram (both pairs of opposite sides are parallel), so

$$AE = DC = 39, \quad CE = DA = 5, \quad EB = AB - AE = 52 - 39 = 13.$$

Step 2: a right triangle. Triangle CEB has sides $CE = 5$, $BC = 12$, $EB = 13$. Since $5^2 + 12^2 = 25 + 144 = 169 = 13^2$, it is a right triangle with the right angle at C , opposite the longest side $\overline{EB}$.

Step 3: the height. The height of the trapezoid is the distance from C to line AB , which is also the height of $\triangle CEB$ from C to $\overline{EB}$. Compute the area of $\triangle CEB$ two ways: using the legs, it is $\frac{1}{2} \cdot 5 \cdot 12 = 30$; using base EB , it is $\frac{1}{2} \cdot 13 \cdot h$. So $13h = 60$ and $h = \frac{60}{13}$.

Step 4: the area. Writing $[\cdot]$ for area,

$$[ABCD] = \frac{1}{2}(52 + 39) \cdot \frac{60}{13} = \frac{1}{2} \cdot 91 \cdot \frac{60}{13} = \frac{1}{2} \cdot 7 \cdot 60 = 210.$$

Another way. Extend legs $\overline{AD}$ and $\overline{BC}$ to meet at P , beyond D and C . Since $\overline{DC} \parallel \overline{AB}$, $\triangle PDC \sim \triangle PAB$ with ratio $\frac{39}{52} = \frac{3}{4}$. From $\frac{PD}{PD+5} = \frac{3}{4}$ we get $PD = 15$, and from $\frac{PC}{PC+12} = \frac{3}{4}$ we get $PC = 36$; so $PA = 20$ and $PB = 48$. Now $15 : 36 : 39 = 3 \cdot (5 : 12 : 13)$, so the angle at P is a right angle, and

$$[ABCD] = [PAB] - [PDC] = \frac{1}{2} \cdot 20 \cdot 48 - \frac{1}{2} \cdot 15 \cdot 36 = 480 - 270 = 210.$$

Watch out. In the figure $\overline{DA}$ looks almost vertical, but it is slanted: the height is $\frac{60}{13} \approx 4.6$, not 5. Using 5 as the height gives 227.5, which is not a choice. Likewise, the right angle of $\triangle CEB$ is at C , not at B , so 12 is not the height either.

Answer: (C) 210