
AMC 10

SET 4

PRACTICE EXAMINATION

SOLUTIONS

[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 25 problems are taken from past AMC 10 contests (2000–2019) and appear in their original wording, rearranged into a new paper. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	E	2009 AMC 10A #1	Algebra
2	A	2019 AMC 10A #2	Number Theory
3	E	2013 AMC 10A #3	Geometry
4	B	2010 AMC 10A #4	Algebra
5	D	2018 AMC 10B #5	Counting & Probability
6	B	2005 AMC 10A #6	Algebra
7	C	2001 AMC 10 #7	Algebra
8	E	2014 AMC 10B #8	Algebra
9	A	2019 AMC 10B #9	Algebra
10	B	2014 AMC 10A #10	Algebra
11	C	2006 AMC 10A #11	Geometry
12	A	2016 AMC 10A #12	Counting & Probability
13	C	2010 AMC 10B #13	Algebra
14	C	2015 AMC 10A #14	Geometry
15	E	2000 AMC 10 #15	Algebra
16	D	2004 AMC 10A #16	Counting & Probability
17	C	2012 AMC 10B #17	Geometry
18	E	2016 AMC 10B #18	Number Theory
19	E	2018 AMC 10A #19	Counting & Probability
20	D	2004 AMC 10B #20	Geometry
21	C	2012 AMC 10A #21	Geometry
22	B	2003 AMC 10A #22	Geometry
23	C	2009 AMC 10B #23	Counting & Probability
24	C	2017 AMC 10A #24	Algebra
25	D	2007 AMC 10A #25	Number Theory

1 Algebra

Source: 2009 AMC 10A #1

One can holds 12 ounces of soda. What is the minimum number of cans needed to provide a gallon (128 ounces) of soda?

- (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Key idea. Divide the total by the size of one can, then round *up*: a partly filled need still requires a whole can.

Each can holds 12 ounces, and we need 128 ounces. Dividing,

$$\frac{128}{12} = 10\frac{2}{3}.$$

So 10 cans are not quite enough: they hold only $10 \cdot 12 = 120$ ounces, which is 8 ounces short.

Eleven cans hold $11 \cdot 12 = 132$ ounces, which is at least 128. So the minimum is 11 cans.

Watch out. Just dropping the fraction in $10\frac{2}{3}$ (rounding *down*) gives 10, choice (D). But 10 cans leave you 8 ounces short of a gallon — when you need “enough”, always round up.

Answer: (E) 11**2** Number Theory

Source: 2019 AMC 10A #2

What is the hundreds digit of $(20! - 15!)$?

- (A) 0 (B) 1 (C) 2 (D) 4 (E) 5

Key idea. You do not need to multiply anything out. The hundreds digit of a number is 0 whenever the number is a multiple of 1000, so ask how many factors of 10 each factorial contains.

Step 1. $15! = 1 \cdot 2 \cdot 3 \cdots 15$ contains the factors 5, 10 and 15, each giving one factor of 5, so $15!$ has three factors of 5. It certainly has at least three factors of 2 (from 2, 4, 6, ...). Hence $15!$ is divisible by $2^3 \cdot 5^3 = 1000$.

Step 2. $20!$ is $15!$ multiplied by $16 \cdot 17 \cdot 18 \cdot 19 \cdot 20$, so it is also a multiple of 1000.

Step 3. The difference of two multiples of 1000 is a multiple of 1000, so $20! - 15!$ ends in 000. Its hundreds digit is 0.

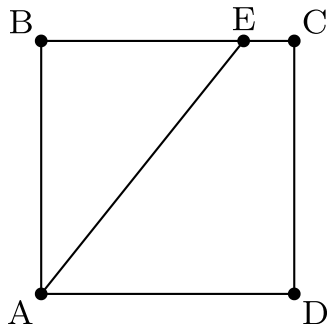
(Indeed, $20! - 15! = 2\,432\,900\,700\,502\,272\,000$.)

Answer: (A) 0

3 Geometry

Source: 2013 AMC 10A #3

Square $ABCD$ has side length 10. Point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40. What is BE ?



- (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Key idea. $\triangle ABE$ has a right angle at B (a corner of the square), so its two legs AB and BE serve as base and height.

Since E lies on side $\overline{BC}$, the segments $\overline{AB}$ and $\overline{BE}$ meet at the square's right angle at B . Take $\overline{BE}$ as the base and $\overline{AB}$, of length 10, as the height:

$$40 = \frac{1}{2} \cdot AB \cdot BE = \frac{1}{2} \cdot 10 \cdot BE = 5BE.$$

Dividing by 5 gives $BE = 8$.

Watch out. Forgetting the $\frac{1}{2}$ in the triangle-area formula gives $40 = 10 \cdot BE$, so $BE = 4$, which is choice (A).

Answer: (E) 8

4 Algebra

Source: 2010 AMC 10A #4

A book that is to be recorded onto compact discs takes 412 minutes to read aloud. Each disc can hold up to 56 minutes of reading. Assume that the smallest possible number of discs is used and that each disc contains the same length of reading. How many minutes of reading will each disc contain?

- (A) 50.2 (B) 51.5 (C) 52.4 (D) 53.8 (E) 55.2

Key idea. First find how many discs are needed (round up), then split the 412 minutes equally among them.

Step 1. Seven discs hold at most $7 \cdot 56 = 392$ minutes, which is less than 412, so 7 discs are not enough. Eight discs hold up to $8 \cdot 56 = 448$ minutes, which is enough. So the smallest number of discs is 8.

Step 2. Sharing 412 minutes equally among 8 discs gives

$$412 \div 8 = 51.5 \text{ minutes per disc,}$$

which is below the 56-minute limit, as it must be.

Watch out. Since $412 \div 56 \approx 7.36$, it is tempting to use 7 discs. But then each disc would need $412 \div 7 \approx 58.9$ minutes, more than a disc can hold — that value is not among the choices, a hint that 7 is wrong.

Answer: (B) 51.5

5 Counting & Probability

Source: 2018 AMC 10B #5

How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

- (A) 128 (B) 192 (C) 224 (D) 240 (E) 256

Key idea. “At least one” is a signal to count the complement: count all subsets, then subtract the ones with *no* prime.

Step 1. The set has 8 elements. Each element is either in or out of a subset, so there are $2^8 = 256$ subsets in all.

Step 2. The primes in the set are 2, 3, 5, 7. A subset contains no prime exactly when it uses only the non-primes $\{4, 6, 8, 9\}$. That set has 4 elements, so it has $2^4 = 16$ subsets (including the empty set, which indeed contains no prime).

Step 3. Subsets with at least one prime: $256 - 16 = 240$.

Watch out. Forgetting that 2 is prime makes the non-primes $\{2, 4, 6, 8, 9\}$, giving $2^8 - 2^5 = 256 - 32 = 224$, which is choice (C).

Answer: (D) 240

6 Algebra

Source: 2005 AMC 10A #6

The average (mean) of 20 numbers is 30, and the average of 30 other numbers is 20. What is the average of all 50 numbers?

- (A) 23 (B) 24 (C) 25 (D) 26 (E) 27

Key idea. An average hides a total. Recover each group's total (average $\times$ count), add the totals, then divide by the total count.

Step 1. The 20 numbers with average 30 add up to $20 \cdot 30 = 600$.

Step 2. The 30 numbers with average 20 add up to $30 \cdot 20 = 600$.

Step 3. All 50 numbers add up to $600 + 600 = 1200$, so their average is

$$\frac{1200}{50} = 24.$$

This is closer to 20 than to 30, which makes sense: there are more numbers in the group whose average is 20.

Watch out. Averaging the two averages, $\frac{30+20}{2} = 25$, gives choice (C). That only works when the two groups have the *same* size; here the groups have 20 and 30 numbers.

Answer: (B) 24**7** Algebra

Source: 2001 AMC 10 #7

When the decimal point of a certain positive decimal number is moved four places to the right, the new number is four times the reciprocal of the original number. What is the original number?

- (A) 0.0002 (B) 0.002 (C) 0.02 (D) 0.2 (E) 2

Key idea. Translate the words into an equation: moving the decimal point four places to the right multiplies the number by 10 000.

Step 1. Call the original number x , with $x > 0$. The new number is $10\,000x$, and four times the reciprocal of x is $4 \cdot \frac{1}{x} = \frac{4}{x}$. So

$$10\,000x = \frac{4}{x}.$$

Step 2. Multiply both sides by x and divide by 10 000:

$$x^2 = \frac{4}{10\,000}.$$

Since x is positive, $x = \frac{2}{100} = 0.02$.

Check. Moving the decimal point of 0.02 four places right gives 200, and $4 \cdot \frac{1}{0.02} = 4 \cdot 50 = 200$. They agree.

Answer: (C) 0.02

8 Algebra

Source: 2014 AMC 10B #8

A truck travels $\frac{b}{6}$ feet every t seconds. There are 3 feet in a yard. How many yards does the truck travel in 3 minutes?

- (A) $\frac{b}{1080t}$ (B) $\frac{30t}{b}$ (C) $\frac{30b}{t}$ (D) $\frac{10t}{b}$ (E) $\frac{10b}{t}$

Key idea. Convert everything to the units the question asks for — yards and seconds — and then find the rate per second.

Step 1. Feet to yards: $\frac{b}{6}$ feet is $\frac{b}{6} \div 3 = \frac{b}{18}$ yards. So the truck goes $\frac{b}{18}$ yards every t seconds.

Step 2. Rate per second: dividing by t , the truck goes $\frac{b}{18t}$ yards each second.

Step 3. Three minutes is $3 \cdot 60 = 180$ seconds, so the distance is

$$\frac{b}{18t} \cdot 180 = \frac{180b}{18t} = \frac{10b}{t} \text{ yards.}$$

Watch out. Forgetting to convert feet to yards gives $\frac{b}{6t} \cdot 180 = \frac{30b}{t}$, which is choice (C). Always check that your answer is in the unit the question asks for.

Answer: (E) $\frac{10b}{t}$

9 Algebra

Source: 2019 AMC 10B #9

The function f is defined by

$$f(x) = \lfloor |x| \rfloor - \lfloor x \rfloor$$

for all real numbers x , where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r . What is the range of f ?

- (A) $\{-1, 0\}$ (B) The set of nonpositive integers
 (C) $\{-1, 0, 1\}$ (D) $\{0\}$
 (E) The set of nonnegative integers

Key idea. The two terms differ only in the *order* of “take the absolute value” and “round down”. Those two operations commute for $x \geq 0$, so the interesting case is negative x — split into cases.

Case 1: $x \geq 0$. Then $|x| = x$ and $\lfloor x \rfloor \geq 0$, so both terms equal $\lfloor x \rfloor$ and $f(x) = 0$.

Case 2: x a negative integer. Then $\lfloor x \rfloor = x$, and both terms equal $|x|$, so $f(x) = 0$.

Case 3: x negative, not an integer. Write $x = -k - t$ with k a nonnegative integer and $0 < t < 1$. Then $|x| = k + t$, so $\lfloor |x| \rfloor = k$. But rounding x down moves it further from zero: $\lfloor x \rfloor = -k - 1$, so $\lfloor |x| \rfloor = k + 1$. Hence $f(x) = k - (k + 1) = -1$.

For example, $x = -2.5$: $\lfloor 2.5 \rfloor = 2$ and $\lfloor -2.5 \rfloor = -3$, so $f(-2.5) = -1$.

Both values 0 and -1 occur, and no other value is possible, so the range is $\{-1, 0\}$.

Watch out. Testing only positive numbers (or only integers) gives 0 every time and suggests choice (D). Remember that $\lfloor -2.5 \rfloor = -3$, not -2 .

Answer: (A) $\{-1, 0\}$

10 Algebra

Source: 2014 AMC 10A #10

Five positive consecutive integers starting with a have average b . What is the average of 5 consecutive integers that start with b ?

- (A) $a + 3$ (B) $a + 4$ (C) $a + 5$ (D) $a + 6$ (E) $a + 7$

Key idea. For five consecutive integers the average is the middle one, which is the first one plus 2.

Step 1. The integers $a, a + 1, a + 2, a + 3, a + 4$ have sum $5a + 10$, so their average is

$$b = \frac{5a + 10}{5} = a + 2.$$

Step 2. By the same reasoning, the five consecutive integers starting with b have average $b + 2$.

Step 3. Substituting $b = a + 2$ gives $b + 2 = (a + 2) + 2 = a + 4$.

Check. With $a = 1$: the numbers 1, 2, 3, 4, 5 have average $b = 3$, and 3, 4, 5, 6, 7 have average $5 = 1 + 4$.

Answer: (B) $a + 4$

11 Geometry

Source: 2006 AMC 10A #11

Which of the following describes the graph of the equation $(x + y)^2 = x^2 + y^2$?

- (A) the empty set (B) one point (C) two lines
(D) a circle (E) the entire plane

Key idea. Expand the square. The equation compares $(x + y)^2$ with $x^2 + y^2$, and the only difference between them is the middle term $2xy$ — so everything hinges on when that term vanishes.

Step 1. Expanding the left side,

$$x^2 + 2xy + y^2 = x^2 + y^2.$$

Step 2. Subtract $x^2 + y^2$ from both sides: $2xy = 0$, that is, $xy = 0$.

Step 3. A product is zero exactly when one of its factors is zero, so the equation holds exactly when $x = 0$ or $y = 0$. The points with $x = 0$ form the y -axis and the points with $y = 0$ form the x -axis.

So the graph is the two coordinate axes: two lines (crossing at the origin). For example, $(0, 5)$ works since $25 = 0 + 25$, but $(1, 1)$ fails since $4 \neq 2$.

Watch out. Believing that $(x + y)^2 = x^2 + y^2$ for all x, y (forgetting the $2xy$) makes the equation look true everywhere, choice (E). The test point $(1, 1)$ shows it is not.

Answer: (C) two lines

12 Counting & Probability

Source: 2016 AMC 10A #12

Three distinct integers are selected at random between 1 and 2016, inclusive. Which of the following is a correct statement about the probability p that the product of the three integers is odd?

- (A) $p < \frac{1}{8}$ (B) $p = \frac{1}{8}$ (C) $\frac{1}{8} < p < \frac{1}{3}$ (D) $p = \frac{1}{3}$ (E) $p > \frac{1}{3}$

Key idea. A product is odd only when *every* factor is odd. The word *distinct* matters: the integers are chosen without replacement, so compare the exact probability with the “independent coin flips” value $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

Step 1. From 1 to 2016 there are 1008 odd and 1008 even integers. The product of the three chosen integers is odd exactly when all three are odd.

Step 2. Pick the integers one at a time. The first is odd with probability $\frac{1008}{2016}$. Given that, 1007 odd integers remain among 2015, and then 1006 odd among 2014. So

$$p = \frac{1008}{2016} \cdot \frac{1007}{2015} \cdot \frac{1006}{2014}.$$

Step 3. Compare each factor with $\frac{1}{2}$. The first equals $\frac{1}{2}$. The second is less than $\frac{1}{2}$ because $2 \cdot 1007 = 2014 < 2015$. The third is less than $\frac{1}{2}$ because $2 \cdot 1006 = 2012 < 2014$.

Therefore $p < \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. (In fact $p = \frac{503}{4030} \approx 0.1248$, just under 0.125.)

Watch out. Treating the three choices as independent, each odd with probability $\frac{1}{2}$, gives $p = \frac{1}{8}$, choice (B). Because the integers are *distinct*, each odd pick leaves slightly fewer odd numbers behind, so p is a little smaller.

Answer: (A) $p < \frac{1}{8}$

13 Algebra

Source: 2010 AMC 10B #13

What is the sum of all the solutions of

$$x = |2x - |60 - 2x||?$$

(A) 32

(B) 60

(C) 92

(D) 120

(E) 124

Key idea. Nested absolute values are removed one layer at a time, each layer giving two cases. Also notice the left side is x and the right side is an absolute value, so every solution must satisfy $x \geq 0$ — that lets you throw out bad candidates at the end.

Step 1: the outer absolute value. $x = |A|$ requires $A = x$ or $A = -x$, where $A = 2x - |60 - 2x|$:

$$2x - |60 - 2x| = x \quad \text{or} \quad 2x - |60 - 2x| = -x,$$

which rearrange to

$$|60 - 2x| = x \quad \text{or} \quad |60 - 2x| = 3x.$$

Step 2: the first equation, $|60 - 2x| = x$. Either $60 - 2x = x$, giving $3x = 60$ and $x = 20$; or $60 - 2x = -x$, giving $x = 60$.

Step 3: the second equation, $|60 - 2x| = 3x$. Either $60 - 2x = 3x$, giving $5x = 60$ and $x = 12$; or $60 - 2x = -3x$, giving $x = -60$.

Step 4: check. Since the right side of the original equation is an absolute value, x must be nonnegative, so $x = -60$ is rejected. (Directly: the right side at $x = -60$ is $|-120 - 180| = 300 \neq -60$.) The other three work:

$$x = 12 : |24 - 36| = 12,$$

$$x = 20 : |40 - 20| = 20,$$

$$x = 60 : |120 - 60| = 60.$$

The sum of the solutions is $12 + 20 + 60 = 92$.

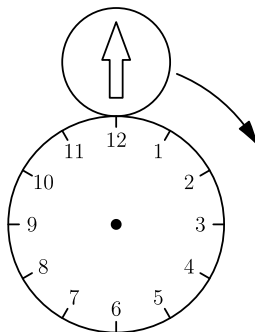
Watch out. Keeping the extraneous root $x = -60$ gives $12 + 20 + 60 - 60 = 32$, choice (A). Always substitute candidates back into an absolute-value equation.

Answer: (C) 92

14 Geometry

Source: 2015 AMC 10A #14

The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock face at 12 o'clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. Let the disk roll clockwise around the clock face. At what point on the clock face will the disk be tangent when the arrow is next pointing in the upward vertical direction?



- (A) 2 o'clock (B) 3 o'clock (C) 4 o'clock (D) 6 o'clock (E) 8 o'clock

Key idea. The arrow turns for two separate reasons: the disk *spins* because it rolls along the clock's edge, and it also *travels around* the clock, which by itself turns it. The phrase “externally tangent ... roll around” is the signal that both effects add up.

Step 1: name the motion. Let θ be the angle, measured clockwise at the clock's center, through which the point of tangency has moved from 12 o'clock. Each hour mark is $\frac{360^\circ}{12} = 30^\circ$ apart.

Step 2: the rolling part. Rolling without slipping means the disk's rim and the clock's rim pass the same length. The tangency point has moved along an arc of length 20θ of the clock. That same length of the disk's rim has rolled over it, which turns the disk by $\frac{20\theta}{10} = 2\theta$ relative to the line joining the two centers.

Step 3: the travelling part. The line joining the two centers has itself turned clockwise through θ . Hence the arrow's total clockwise turn is

$$2\theta + \theta = 3\theta.$$

(Equivalently: the disk's center moves along a circle of radius $20 + 10 = 30$, so its path has length 30θ , and $\frac{30\theta}{10} = 3\theta$.)

Step 4: when is the arrow up again? The arrow points straight up again the first time its total turn is a full turn: $3\theta = 360^\circ$, so $\theta = 120^\circ$. That is $\frac{120^\circ}{30^\circ} = 4$ hour marks clockwise from 12, namely 4 o'clock.

Watch out. Counting only the rolling ($2\theta = 360^\circ$, so $\theta = 180^\circ$) gives 6 o'clock, choice (D). This misses the extra turn the disk gets from travelling around the outside of the clock — the same reason a coin rolled around an equal coin makes *two* turns, not one.

Answer: (C) 4 o'clock

15 Algebra

Source: 2000 AMC 10 #15

Two non-zero real numbers, a and b , satisfy $ab = a - b$. Find a possible value of

$$\frac{a}{b} + \frac{b}{a} - ab.$$

- (A) -2 (B) $-\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) 2

Key idea. The condition $ab = a - b$ is a substitution waiting to happen. Putting the expression over the common denominator ab produces $a^2 + b^2$, and $(ab)^2 = (a - b)^2$ also contains $a^2 + b^2$ — so they cancel.

Step 1: one fraction. Over the common denominator ab ,

$$\frac{a}{b} + \frac{b}{a} - ab = \frac{a^2 + b^2 - (ab)^2}{ab}.$$

Step 2: substitute. Replace ab by $a - b$ in the numerator only:

$$(ab)^2 = (a - b)^2 = a^2 - 2ab + b^2.$$

So the numerator is $a^2 + b^2 - (a^2 - 2ab + b^2) = 2ab$.

Step 3: simplify. The expression equals $\frac{2ab}{ab} = 2$, which is allowed because a and b are non-zero.

Another way. Since the question asks only for a *possible* value, one example is enough. From $ab = a - b$ we get $b(a + 1) = a$, so $b = \frac{a}{a+1}$. Take $a = 1$, $b = \frac{1}{2}$: then $ab = \frac{1}{2} = a - b$, and

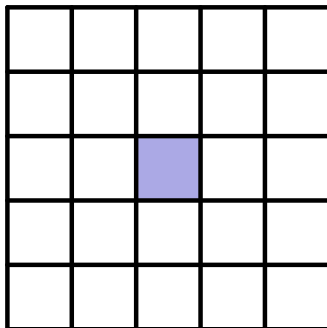
$$\frac{1}{1/2} + \frac{1/2}{1} - \frac{1}{2} = 2 + \frac{1}{2} - \frac{1}{2} = 2.$$

Answer: (E) 2

16 Counting & Probability

Source: 2004 AMC 10A #16

The 5×5 grid shown contains a collection of squares with sizes from 1×1 to 5×5 . How many of these squares contain the shaded center square?



- (A) 12 (B) 15 (C) 17 (D) 19 (E) 20

Key idea. Sort the squares by size and, for each size, count how many positions cover the center. “Contains the center” is really two separate conditions — the square’s rows must include row 3 and its columns must include column 3 — so count the row positions, count the column positions, and multiply.

Step 1. Number the rows 1 to 5; the shaded square is in row 3. A $k \times k$ square occupies k consecutive rows, and we need row 3 among them.

- $k = 1$: only rows $\{3\}$ — 1 choice.
- $k = 2$: rows $\{2, 3\}$ or $\{3, 4\}$ — 2 choices.
- $k = 3$: rows $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$ — 3 choices.
- $k = 4$: rows $\{1, 2, 3, 4\}$ or $\{2, 3, 4, 5\}$ — 2 choices.
- $k = 5$: all five rows — 1 choice.

Step 2. Columns work exactly the same way, and the row choice and column choice are independent. So the number of $k \times k$ squares containing the center is the row count squared:

$$1^2 + 2^2 + 3^2 + 2^2 + 1^2 = 1 + 4 + 9 + 4 + 1 = 19.$$

In words: the center square itself, four 2×2 squares (one at each corner of the center), all nine 3×3 squares, all four 4×4 squares, and the whole grid.

Watch out. The 4×4 squares are the easiest to overlook: none of them is centered on the shaded cell, yet each misses only one outer row and one outer column, so all four contain the center. Leaving them out (or leaving out the four 2×2 squares) gives $19 - 4 = 15$, choice (B).

Answer: (D) 19

17 Geometry

Source: 2012 AMC 10B #17

Jesse cuts a circular paper disk of radius 12 along two radii to form two sectors, the smaller having a central angle of 120 degrees. He makes two circular cones, using each sector to form the lateral surface of a cone. What is the ratio of the volume of the smaller cone to that of the larger?

- (A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{\sqrt{10}}{10}$ (D) $\frac{\sqrt{5}}{6}$ (E) $\frac{\sqrt{10}}{5}$

Key idea. When a sector is rolled into a cone, two things carry over: the sector's radius becomes the cone's *slant height*, and the sector's arc becomes the *circumference of the base*. The statement gives you exactly a radius and an angle, so these two facts are all you need; then the Pythagorean theorem gives each height.

Step 1: the sectors. The two sectors together make the whole disk, so the larger one has central angle $360^\circ - 120^\circ = 240^\circ$. Both have radius 12, so both cones have slant height 12.

Step 2: base radii. The full circle has circumference $2\pi \cdot 12 = 24\pi$. The 120° arc is $\frac{1}{3}$ of it, 8π ; setting $2\pi r = 8\pi$ gives base radius $r = 4$. The 240° arc is $\frac{2}{3}$ of it, 16π , so the larger cone has base radius 8.

Step 3: heights. The height, base radius and slant height form a right triangle:

$$h_{\text{small}} = \sqrt{12^2 - 4^2} = \sqrt{128} = 8\sqrt{2}, \quad h_{\text{large}} = \sqrt{12^2 - 8^2} = \sqrt{80} = 4\sqrt{5}.$$

Step 4: volumes. With $V = \frac{1}{3}\pi r^2 h$, the factor $\frac{1}{3}\pi$ cancels in the ratio:

$$\frac{V_{\text{small}}}{V_{\text{large}}} = \frac{4^2 \cdot 8\sqrt{2}}{8^2 \cdot 4\sqrt{5}} = \frac{128\sqrt{2}}{256\sqrt{5}} = \frac{\sqrt{2}}{2\sqrt{5}} = \frac{\sqrt{2} \cdot \sqrt{5}}{2 \cdot 5} = \frac{\sqrt{10}}{10}.$$

Watch out. The two cones are *not* similar: they share the slant height 12, so the one with the narrower base is the taller one. Treating them as similar with radius ratio $\frac{4}{8} = \frac{1}{2}$ and cubing gives $\frac{1}{8}$, choice (A); comparing only the base areas, $\left(\frac{1}{2}\right)^2$, gives $\frac{1}{4}$, choice (B).

Answer: (C) $\frac{\sqrt{10}}{10}$

18 Number Theory

Source: 2016 AMC 10B #18

In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

- (A) 1 (B) 3 (C) 5 (D) 6 (E) 7

Key idea. A run of consecutive integers is an arithmetic sequence, so its sum has a formula: (number of terms) $\times$ (average). Writing that formula out turns “sum equals 345” into “a product of two integers equals 690”, and counting factorizations of a number is something you know how to do.

Step 1: set up. Let the sequence have $n \geq 2$ terms and first term $a \geq 1$, so the terms are $a, a+1, \dots, a+n-1$. Their sum is n times the average of the first and last terms:

$$n \cdot \frac{a + (a + n - 1)}{2} = 345 \implies n(2a + n - 1) = 690.$$

Step 2: what the two factors must look like. Call the second factor $m = 2a + n - 1$. Then:

- $m - n = 2a - 1$ is odd, so n and m have opposite parity;
- $a \geq 1$ means $m - n \geq 1$, so $n < m$;
- conversely, any factorization $690 = n \cdot m$ with $n < m$ of opposite parity gives a valid $a = \frac{m-n+1}{2}$, a positive integer.

Step 3: count. $690 = 2 \cdot 3 \cdot 5 \cdot 23$ has $2^4 = 16$ divisors, so 8 factor pairs $n < m$. Since 690 contains exactly one factor of 2, in every pair one factor is even and the other odd — all 8 pairs pass the parity test. Only the pair $n = 1, m = 690$ is ruled out, because it is the one-term “sum” 345 itself. That leaves 7.

Check. The seven sums, by length n :

n	2	3	5	6	10	15	23
terms	172–173	114–116	67–71	55–60	30–39	16–30	4–26

For instance $4 + 5 + \dots + 26$ has 23 terms with average 15, and $23 \cdot 15 = 345$.

Watch out. The “middle term” shortcut — 345 divided by an odd length n gives the middle term — finds only the odd lengths 3, 5, 15, 23. Even lengths such as $172 + 173$ have no middle term but still count, which is why the factorization of $2 \cdot 345 = 690$, not 345, is the right thing to look at.

Answer: (E) 7

19 Counting & Probability

Source: 2018 AMC 10A #19

A number m is randomly selected from the set

$$\{11, 13, 15, 17, 19\},$$

and a number n is randomly selected from

$$\{1999, 2000, 2001, \dots, 2018\}.$$

What is the probability that m^n has a units digit of 1?

- (A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{3}{10}$ (D) $\frac{7}{20}$ (E) $\frac{2}{5}$

Key idea. The units digit of a power depends only on the units digit of the base, and those units digits repeat in short cycles (length 1, 2 or 4). Exponents around 2000 are the signal: you are meant to look at the cycle, not the number. So replace m by its last digit and n by its position in the cycle.

Step 1: count the pairs. There are 5 choices of m and $2018 - 1999 + 1 = 20$ choices of n , so $5 \cdot 20 = 100$ equally likely pairs. We count the pairs where m^n ends in 1.

Step 2: the cycles. Units digits of successive powers:

- base ending in 1: 1, 1, 1, ... — always 1;
- base ending in 3: 3, 9, 7, 1, 3, ... — ends in 1 exactly when $4 \mid n$;
- base ending in 5: always 5 — never 1;
- base ending in 7: 7, 9, 3, 1, 7, ... — ends in 1 exactly when $4 \mid n$;
- base ending in 9: 9, 1, 9, 1, ... — ends in 1 exactly when n is even.

Step 3: the exponents. The 20 consecutive values 1999, ..., 2018 contain 10 even numbers, and 5 multiples of 4 (namely 2000, 2004, 2008, 2012, 2016).

Step 4: add up. Favorable pairs:

$$\underbrace{20}_{m=11} + \underbrace{5}_{m=13} + \underbrace{0}_{m=15} + \underbrace{5}_{m=17} + \underbrace{10}_{m=19} = 40,$$

so the probability is $\frac{40}{100} = \frac{2}{5}$.

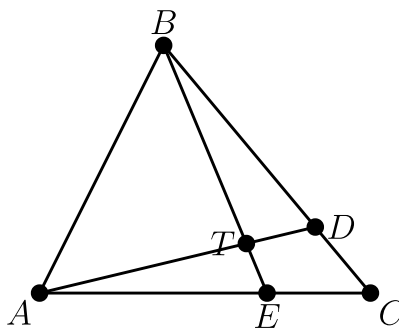
Watch out. Do not assume every odd digit has a cycle of length 4. The digit 9 cycles with length 2 ($9^2 = 81$), so 19^n ends in 1 for every even n — 10 exponents, not 5. Treating 9 like 3 and 7 gives $\frac{35}{100} = \frac{7}{20}$, choice (D).

Answer: (E) $\frac{2}{5}$

20 Geometry

Source: 2004 AMC 10B #20

In $\triangle ABC$ points D and E lie on $\overline{BC}$ and $\overline{AC}$, respectively. If $\overline{AD}$ and $\overline{BE}$ intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$?



- (A) $\frac{1}{8}$ (B) $\frac{2}{9}$ (C) $\frac{3}{10}$ (D) $\frac{4}{11}$ (E) $\frac{5}{12}$

Key idea. Ratios along two crossing segments are easiest with *mass points*: put weights on the vertices so that each segment balances at T .

Step 1. On $\overline{AD}$, $AT : TD = 3 : 1$. A lever balances when weight $\times$ distance is equal, so give A weight 1 and D weight 3.

Step 2. On $\overline{BE}$, $BT : TE = 4 : 1$, so the weight at E must be 4 times the weight at B . Call the weight at B b ; then E has weight $4b$.

Step 3. D sits on $\overline{BC}$, so its weight 3 is shared as $b + c = 3$, where c is the weight at C . Likewise E sits on $\overline{AC}$: $1 + c = 4b$.

Solving, $b + c = 3$ and $c = 4b - 1$ give $5b = 4$, so $b = \frac{4}{5}$ and $c = \frac{11}{5}$.

Step 4. On $\overline{BC}$ the point D is closer to the heavier end: $\frac{CD}{BD} = \frac{b}{c} = \frac{4/5}{11/5} = \frac{4}{11}$.

Watch out. The weights are *inversely* proportional to the distances. Writing $\frac{CD}{BD} = \frac{c}{b}$ gives $\frac{11}{4}$, which is not even a choice — a quick sign you flipped it.

Answer: (D) $\frac{4}{11}$

21 Geometry

Source: 2012 AMC 10A #21

Let points $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 2, 0)$, and $D = (0, 0, 3)$. Points E , F , G , and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of $EFGH$?

- (A) $\sqrt{2}$ (B) $\frac{2\sqrt{5}}{3}$ (C) $\frac{3\sqrt{5}}{4}$ (D) $\sqrt{3}$ (E) $\frac{2\sqrt{7}}{3}$

Key idea. Four *midpoints* should make you think of *midsegments*: the segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long. Each side of $EFGH$ is a midsegment of one face of the tetrahedron $ABCD$, so each is half of an edge you already know — and that works in 3D just as in the plane, because each face is a flat triangle.

Step 1: identify the midsegments.

- In $\triangle ABD$, F and E are the midpoints of $\overline{AB}$ and $\overline{BD}$, so $\overline{FE} \parallel \overline{AD}$ and $FE = \frac{1}{2}AD$.
- In $\triangle ACD$, G and H are the midpoints of $\overline{AC}$ and $\overline{DC}$, so $\overline{GH} \parallel \overline{AD}$ and $GH = \frac{1}{2}AD$.
- In $\triangle ABC$, F and G are the midpoints of $\overline{AB}$ and $\overline{AC}$, so $\overline{FG} \parallel \overline{BC}$ and $FG = \frac{1}{2}BC$.

Step 2: the shape. $\overline{FE}$ and $\overline{GH}$ are parallel, equally long, and point the same way (from the base ABC up toward D), so $EFGH$ is a parallelogram — in particular, it is flat. Its sides are parallel to $\overline{AD}$ and $\overline{BC}$. Now $\overline{AD}$ runs along the z -axis while $\overline{BC}$ lies in the xy -plane, so they are perpendicular. A parallelogram with perpendicular adjacent sides is a rectangle.

Step 3: the side lengths. $AD = 3$, so $FE = \frac{3}{2}$. $BC = \sqrt{1^2 + 2^2} = \sqrt{5}$, so $FG = \frac{\sqrt{5}}{2}$.

Step 4: the area.

$$FE \cdot FG = \frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \frac{3\sqrt{5}}{4}.$$

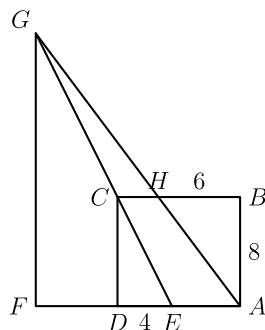
Another way. Coordinates work directly. Averaging endpoints, $E = (\frac{1}{2}, 0, \frac{3}{2})$, $F = (\frac{1}{2}, 0, 0)$, $G = (0, 1, 0)$, $H = (0, 1, \frac{3}{2})$. Then $\overrightarrow{FE} = (0, 0, \frac{3}{2})$ and $\overrightarrow{GH} = (0, 0, \frac{3}{2})$ are equal, and $\overrightarrow{FG} = (-\frac{1}{2}, 1, 0)$ has dot product 0 with $\overrightarrow{FE}$. So $EFGH$ is a rectangle with sides $\frac{3}{2}$ and $\sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5}}{2}$, area $\frac{3\sqrt{5}}{4}$.

Answer: (C) $\frac{3\sqrt{5}}{4}$

22 Geometry

Source: 2003 AMC 10A #22

In rectangle $ABCD$, we have $AB = 8$, $BC = 9$, H is on $\overline{BC}$ with $BH = 6$, E is on $\overline{AD}$ with $DE = 4$, line EC intersects line AH at G , and F is on line $\overline{AD}$ with $\overline{GF} \perp \overline{AF}$. Find the length $\overline{GF}$.



- (A) 16 (B) 20 (C) 24 (D) 28 (E) 30

Key idea. Lines GE and GA both start at G and cross the two *parallel* sides $\overline{BC}$ and $\overline{AD}$ of the rectangle. Two parallel lines cut by two lines through one point always give similar triangles — here $\triangle GCH \sim \triangle GEA$ — and similar triangles have heights in the same ratio as their bases. Since GF is exactly the height of $\triangle GEA$, that ratio is all we need.

Step 1: the two bases. On side $\overline{BC}$ (length 9), $BH = 6$, so $CH = 9 - 6 = 3$. On side $\overline{AD}$ (length 9), $DE = 4$, so $EA = 9 - 4 = 5$.

Step 2: similar triangles. C lies on line GE and H lies on line GA , and $\overline{CH}$ is parallel to $\overline{EA}$ (opposite sides of the rectangle). So $\triangle GCH \sim \triangle GEA$, with ratio $CH : EA = 3 : 5$.

Step 3: the heights. Let $GF = h$; this is the distance from G to line AD , the height of $\triangle GEA$. Line BC is 8 units closer to G , so the height of $\triangle GCH$ is $h - 8$. Heights scale like the bases:

$$\frac{h-8}{h} = \frac{3}{5} \implies 5h - 40 = 3h \implies h = 20.$$

Another way. Coordinates. Put $D = (0, 0)$, $A = (9, 0)$, $B = (9, 8)$, $C = (0, 8)$; then $H = (3, 8)$ and $E = (4, 0)$. Line AH has slope $\frac{8-0}{3-9} = -\frac{4}{3}$, so $y = -\frac{4}{3}(x-9) = -\frac{4}{3}x + 12$. Line EC has slope $\frac{8-0}{0-4} = -2$, so $y = -2x + 8$. Setting them equal: $-\frac{4}{3}x + 12 = -2x + 8$ gives $\frac{2}{3}x = -4$, so $x = -6$ and $y = 20$. Thus $G = (-6, 20)$, and its distance to line AD (the x -axis) is $GF = 20$.

Watch out. Use the pieces of the sides that are actually cut off by the two lines: $\overline{CH}$ and $\overline{EA}$, not the given lengths BH and DE . Comparing $CH = 3$ with $DE = 4$ instead of $EA = 5$ leads to $\frac{h-8}{h} = \frac{3}{4}$, i.e. $h = 32$, which is not among the choices.

Answer: (B) 20

23 Counting & Probability

Source: 2009 AMC 10B #23

Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the start line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture?

- (A) $\frac{1}{16}$ (B) $\frac{1}{8}$ (C) $\frac{3}{16}$ (D) $\frac{1}{4}$ (E) $\frac{5}{16}$

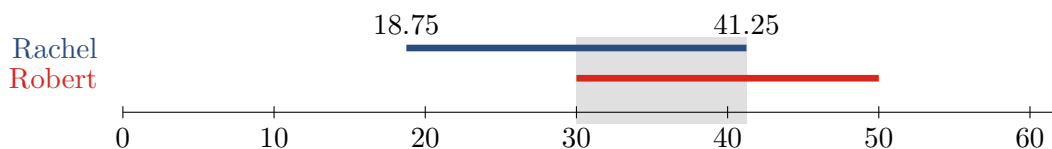
Key idea. The photo time is uniform over one minute, so the probability is (seconds during which *both* are in view) $\div 60$. Turn each runner into a time interval: find where they are at the 10-minute mark, then when they are within the picture. The answer is the overlap of two intervals — positions on a circle become simple arithmetic once you measure them in “seconds of running”.

Step 1: what the picture covers. One-fourth of the track centered on the start line means $\frac{1}{8}$ of a lap on each side of the line. A runner is in the picture when within $\frac{1}{8}$ lap of the line, whichever side. Directions do not matter for this, since the picture is symmetric about the line.

Step 2: Rachel, 90 s per lap. $\frac{1}{8}$ lap takes her $\frac{90}{8} = 11.25$ seconds. At 600 s she has run $\frac{600}{90} = 6\frac{2}{3}$ laps, so she is $\frac{1}{3}$ lap, i.e. 30 seconds, short of the line; she crosses it at 630 s. She is in the picture from $630 - 11.25 = 618.75$ s to $630 + 11.25 = 641.25$ s. Her neighboring crossings are at 540 s and 720 s, too far away to matter.

Step 3: Robert, 80 s per lap. $\frac{1}{8}$ lap takes him $\frac{80}{8} = 10$ seconds. At 600 s he has run $\frac{600}{80} = 7\frac{1}{2}$ laps, so he is 40 seconds from the line; he crosses it at 640 s. He is in the picture from 630 s to 650 s. His neighboring crossings are at 560 s and 720 s, again outside the window.

Step 4: overlap. Measuring seconds after the 10-minute mark:



Both are in the picture from 30 to 41.25 seconds, a stretch of 11.25 seconds. The probability is

$$\frac{11.25}{60} = \frac{45}{240} = \frac{3}{16}.$$

Watch out. The two events are *not* independent: both depend on the same moment the photo is taken. Multiplying Rachel’s chance $\frac{22.5}{60} = \frac{3}{8}$ by Robert’s $\frac{20}{60} = \frac{1}{3}$ gives $\frac{1}{8}$, choice (B). You must intersect the time intervals instead.

Answer: (C) $\frac{3}{16}$

24 Algebra

Source: 2017 AMC 10A #24

For certain real numbers a , b , and c , the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

- (A) -9009 (B) -8008 (C) -7007 (D) -6006 (E) -5005

Key idea. “Every root of g is a root of f ” is a statement about *factors*: it means $g(x)$ divides $f(x)$. Since f has degree 4 and g degree 3, the quotient is a single linear factor $(x - r)$. That turns the problem into matching coefficients — and the problem hands you exactly two known coefficients of f (1 for x^3 , 100 for x), enough to find the two unknowns a and r .

Step 1: factor. Call the roots of g r_1, r_2, r_3 ; they are distinct, and g has leading coefficient 1, so $g(x) = (x - r_1)(x - r_2)(x - r_3)$. Each r_i is a root of f , so each $(x - r_i)$ is a factor of f , and since the r_i are different, their product $g(x)$ divides $f(x)$. Both f and g have leading coefficient 1, so

$$f(x) = g(x)(x - r) \quad \text{for some number } r.$$

Step 2: expand.

$$(x^3 + ax^2 + x + 10)(x - r) = x^4 + (a - r)x^3 + (1 - ar)x^2 + (10 - r)x - 10r.$$

Step 3: match the known coefficients.

- x : $10 - r = 100$, so $r = -90$.
- x^3 : $a - r = 1$, so $a = 1 + r = -89$.

Step 4: evaluate. There is no need to find b and c : just plug $x = 1$ into the factored form.

$$g(1) = 1 + a + 1 + 10 = 12 - 89 = -77, \quad 1 - r = 1 + 90 = 91,$$

$$f(1) = g(1)(1 - r) = (-77)(91) = -7007.$$

Check. Matching the other two coefficients gives $b = 1 - ar = 1 - (-89)(-90) = -8009$ and $c = -10r = 900$. Then directly $f(1) = 1 + 1 - 8009 + 100 + 900 = -7007$. (With $a = -89$, g really does have three distinct roots, near -0.33 , 0.34 and 89 .)

Watch out. Mind the sign of r when expanding $(x - r)$: the coefficient of x is $10 - r$, not $10 + r$, so r comes out *negative*. Taking $r = 90$ instead leads to $a = 91$ and $f(1) = 103 \cdot (-89) = -9167$, which is not a choice — a sign that something went wrong.

Answer: (C) -7007

25 Number Theory

Source: 2007 AMC 10A #25

For each positive integer n , let $S(n)$ denote the sum of the digits of n . For how many values of n is $n + S(n) + S(S(n)) = 2007$?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Key idea. Digit sums are *tiny* compared with n , so n must be just below 2007 — that gives a short search range. And whenever digit sums appear, think of divisibility by 9: a number and its digit sum leave the same remainder mod 9. That cuts the range to a handful of candidates, which we then check one by one.

Step 1: narrow the range. Since $S(n) \geq 1$, we need $n < 2007$, i.e. $n \leq 2006$. Among $1, \dots, 2006$ the largest digit sum is $S(1999) = 28$, so $S(n) \leq 28$. Among $1, \dots, 28$ the largest digit sum is $S(19) = 10$, so $S(S(n)) \leq 10$. Therefore

$$n = 2007 - S(n) - S(S(n)) \geq 2007 - 28 - 10 = 1969.$$

Step 2: use remainders mod 9. A number and its digit sum leave the same remainder when divided by 9. So n , $S(n)$ and $S(S(n))$ all leave the same remainder, and their sum leaves the same remainder as $3n$. Since $2007 = 9 \cdot 223$, we need $3n$ to be a multiple of 9, so n is a multiple of 3.

Step 3: numbers 2000 to 2006. Write $n = 2000 + d$ with $0 \leq d \leq 6$. Then $S(n) = 2 + d$, a single digit, so $S(S(n)) = 2 + d$ too, and

$$(2000 + d) + (2 + d) + (2 + d) = 2004 + 3d = 2007 \implies d = 1.$$

So $n = 2001$ works ($2001 + 3 + 3 = 2007$), and it is the only one in this range.

Step 4: numbers 1969 to 1999. The multiples of 3 here are 1971, 1974, $\dots$, 1998:

n	1971	1974	1977	1980	1983	1986	1989	1992	1995	1998
$S(n)$	18	21	24	18	21	24	27	21	24	27
$S(S(n))$	9	3	6	9	3	6	9	3	6	9
total	1998	1998	2007	2007	2007	2016	2025	2016	2025	2034

Exactly three work: 1977, 1980 and 1983.

Step 5: count. The solutions are 1977, 1980, 1983 and 2001 — four values of n .

Watch out. Do not stop after the numbers in the 1900s. Because the bound only says $n \geq 1969$, the numbers 2001–2006 must be checked as well; their digit sums are very small, and 2001 slips in. Missing it gives 3, choice (C).

Answer: (D) 4