
AMC 10

SET 1

HALF-LENGTH PRACTICE

SOLUTIONS

[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	C	2003 AMC 10B #10	Counting & Probability
2	A	2015 AMC 10B #12	Geometry
3	B	2016 AMC 10A #13	Counting & Probability
4	B	2010 AMC 10B #14	Algebra
5	D	2005 AMC 10B #15	Counting & Probability
6	D	2002 AMC 10B #16	Number Theory
7	C	2011 AMC 10B #17	Geometry
8	A	2004 AMC 10A #18	Algebra
9	C	2008 AMC 10A #19	Geometry
10	C	2014 AMC 10B #20	Algebra

1 Counting & Probability

Source: 2003 AMC 10B #10

Nebraska, the home of the AMC, changed its license plate scheme. Each old license plate consisted of a letter followed by four digits. Each new license plate consists of three letters followed by three digits. By how many times is the number of possible license plates increased?

- (A) $\frac{26}{10}$ (B) $\frac{26^2}{10^2}$ (C) $\frac{26^2}{10}$ (D) $\frac{26^3}{10^3}$ (E) $\frac{26^3}{10^2}$

Key idea. Count each scheme with the multiplication principle, then divide new by old. Leave everything as powers — the choices are written that way.

Each letter slot has 26 options and each digit slot has 10, and the slots are filled independently.

Old plates: one letter and four digits, so $26 \cdot 10^4$ plates.

New plates: three letters and three digits, so $26^3 \cdot 10^3$ plates.

The number of plates is multiplied by

$$\frac{26^3 \cdot 10^3}{26 \cdot 10^4} = \frac{26^{3-1}}{10^{4-3}} = \frac{26^2}{10}.$$

In words: two letter slots were added (a factor of 26^2) and one digit slot was removed (a factor of $\frac{1}{10}$).

Answer: (C) $\frac{26^2}{10}$

2 Geometry

Source: 2015 AMC 10B #12

For how many integers x is the point $(x, -x)$ inside or on the circle of radius 10 centered at $(5, 5)$?

- (A) 11 (B) 12 (C) 13 (D) 14 (E) 15

Key idea. “Inside or on the circle” means “distance to the center is at most 10.” Square both sides so there is no square root to handle, and the condition becomes a simple inequality in x .

Step 1. The squared distance from $(x, -x)$ to $(5, 5)$ is

$$(x - 5)^2 + (-x - 5)^2 = (x - 5)^2 + (x + 5)^2.$$

Expanding, $(x^2 - 10x + 25) + (x^2 + 10x + 25) = 2x^2 + 50$. The $\pm 10x$ terms cancel, which is why the answer will be symmetric about $x = 0$.

Step 2. Inside or on the circle means $2x^2 + 50 \leq 100$, so $2x^2 \leq 50$ and $x^2 \leq 25$. That is, $-5 \leq x \leq 5$.

Step 3. The integers from -5 to 5 are $-5, -4, \dots, 4, 5$: five negative, five positive, and 0, for 11 in all.

Watch out. The two endpoints $x = \pm 5$ give the points $(5, -5)$ and $(-5, 5)$, which lie exactly *on* the circle: for $(5, -5)$ the squared distance is $0^2 + (-10)^2 = 100$. The problem counts them. Dropping them, or counting $5 - (-5) = 10$ integers, gives a number that is not among the choices.

Another way. Drop a perpendicular from the center $(5, 5)$ to the line $y = -x$; its foot is the origin, at distance $\sqrt{50} = 5\sqrt{2}$. By the Pythagorean theorem the chord on $y = -x$ reaches $\sqrt{100 - 50} = 5\sqrt{2}$ on each side of the origin. The point $(x, -x)$ is $|x|\sqrt{2}$ from the origin, so we need $|x|\sqrt{2} \leq 5\sqrt{2}$, i.e. $|x| \leq 5$ — again 11 integers.

Answer: (A) 11

3 Counting & Probability

Source: 2016 AMC 10A #13

Five friends sat in a movie theater in a row containing 5 seats, numbered 1 to 5 from left to right. (The directions “left” and “right” are from the point of view of the people as they sit in the seats.)

During the movie Ada went to the lobby to get some popcorn. When she returned, she found that Bea had moved two seats to the right, Ceci had moved one seat to the left, and Dee and Edie had switched seats, leaving an end seat for Ada. In which seat had Ada been sitting before she got up?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Key idea. The moves are given as *amounts* (two seats right, one seat left), not as positions. Before and after, the same five seats $1, \dots, 5$ are filled, so the seat numbers always add up to 15 — the total of all the moves must be zero.

Step 1. Measure each move as (new seat) $-$ (old seat), with right counting as positive. Bea’s move is $+2$ and Ceci’s is -1 . Dee and Edie trade seats, so if one moves $+k$ the other moves $-k$; together they contribute 0.

Step 2. The sum of the seat numbers is $1 + 2 + 3 + 4 + 5 = 15$ both before and after, so all five moves add to 0:

$$(+2) + (-1) + 0 + (\text{Ada's move}) = 0,$$

which gives Ada’s move $= -1$. Ada ends up one seat to the left of where she started.

Step 3. Ada ends in an end seat, 1 or 5. Ending in seat 5 would mean she started in seat 6, which does not exist. So she ends in seat 1 and started in seat 2.

Check. Ada in 2, Bea in 1, Ceci in 3, Dee and Edie in 4 and 5. Bea moves to 3, Ceci to 2, Dee and Edie swap 4 and 5, and seat 1 is left for Ada. A computer search over all 120 seatings confirms seat 2 is the only possibility.

Watch out. The question asks where Ada sat *before* she got up. Seat 1 is where she sits *after* returning; answering 1 gives choice A.

Answer: (B) 2

4 Algebra

Source: 2010 AMC 10B #14

The average of the numbers $1, 2, 3, \dots, 98, 99$, and x is $100x$. What is x ?

- (A) $\frac{49}{101}$ (B) $\frac{50}{101}$ (C) $\frac{1}{2}$ (D) $\frac{51}{101}$ (E) $\frac{50}{99}$

Key idea. An average is (sum) $\div$ (how many). Write that equation, clear the denominator, and x appears linearly. The only things to get right are the sum $1 + 2 + \dots + 99$ and the count of numbers.

Step 1. The sum of the first n positive integers is $\frac{n(n+1)}{2}$, so

$$1 + 2 + \dots + 99 = \frac{99 \cdot 100}{2} = 4950.$$

Step 2. There are 99 numbers in the list $1, \dots, 99$ plus x itself, so 100 numbers. The average condition is

$$\frac{4950 + x}{100} = 100x.$$

Step 3. Multiply by 100: $4950 + x = 10000x$, so $4950 = 9999x$ and $x = \frac{4950}{9999}$.

Step 4. Both numbers are multiples of 99: $4950 = 99 \cdot 50$ and $9999 = 99 \cdot 101$. Hence $x = \frac{50}{101}$.

Watch out. The average of $1, \dots, 99$ is 50, and it is tempting to set $100x = 50$, giving $x = \frac{1}{2}$ (choice C). But x itself is one of the numbers being averaged, and it drags the average down below 50.

Another way. Estimate. Since x is small, the average is about $\frac{4950}{100} = 49.5$, so $x \approx 0.495$. Now $\frac{50}{101} \approx 0.4950$, while $\frac{49}{101} \approx 0.485$, $\frac{1}{2} = 0.5$, and $\frac{51}{101}$ and $\frac{50}{99}$ are both about 0.505. Only B fits.

Answer: (B) $\frac{50}{101}$

5 Counting & Probability

Source: 2005 AMC 10B #15

An envelope contains eight bills: 2 ones, 2 fives, 2 tens, and 2 twenties. Two bills are drawn at random without replacement. What is the probability that their sum is \$20 or more?

- (A) $\frac{1}{4}$ (B) $\frac{2}{5}$ (C) $\frac{3}{7}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

Key idea. Treat the eight bills as eight different objects (imagine serial numbers), so that every pair of bills is equally likely. Then count the pairs whose sum reaches \$20.

Step 1. The number of ways to choose 2 bills from 8 is $\binom{8}{2} = \frac{8 \cdot 7}{2} = 28$, all equally likely.

Step 2. Sort the good pairs by how many twenties they contain.

- Both twenties: 1 pair (sum \$40).
- Exactly one twenty: 2 choices of twenty times 6 choices of a smaller bill, $2 \cdot 6 = 12$ pairs (sum at least \$21).
- No twenty: the largest possible sum is $10 + 10 = 20$, which counts. That is 1 pair. Any other pair without a twenty, such as $10 + 5$, is at most \$15.

In total $1 + 12 + 1 = 14$ pairs.

Step 3. The probability is $\frac{14}{28} = \frac{1}{2}$.

Watch out. “\$20 or more” includes exactly \$20. Forgetting the pair of tens gives $\frac{13}{28}$, which is not among the choices — a sign that a case is missing.

Another way. Count the complement. A sum below \$20 needs both bills from the six non-twenties, $\binom{6}{2} = 15$ pairs, except the pair of tens: $15 - 1 = 14$ pairs. So the probability of \$20 or more is $1 - \frac{14}{28} = \frac{1}{2}$.

Answer: (D) $\frac{1}{2}$

6 Number Theory

Source: 2002 AMC 10B #16

For how many integers n is $\frac{n}{20-n}$ the square of an integer?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 10

Key idea. The variable sits in both the numerator and the denominator. Rename the denominator $m = 20 - n$ so that n disappears from the top: the fraction becomes $\frac{20}{m} - 1$, and now it is clear that m must divide 20.

Step 1. Let $m = 20 - n$, so $n = 20 - m$ and $m \neq 0$. Then

$$\frac{n}{20-n} = \frac{20-m}{m} = \frac{20}{m} - 1.$$

Step 2. A square of an integer is an integer that is at least 0. For $\frac{20}{m} - 1$ to be an integer, m must divide 20. For it to be at least 0, we need $\frac{20}{m} \geq 1$, so m is positive (a negative m makes $\frac{20}{m} - 1$ negative). The positive divisors of 20 are $m = 1, 2, 4, 5, 10, 20$.

Step 3. The corresponding values of $\frac{20}{m} - 1$ are

$$19, \quad 9, \quad 4, \quad 3, \quad 1, \quad 0.$$

The perfect squares among them are $9 = 3^2$, $4 = 2^2$, $1 = 1^2$, and $0 = 0^2$, from $m = 2, 4, 10, 20$, that is, $n = 18, 16, 10, 0$.

Check. $\frac{18}{2} = 9$, $\frac{16}{4} = 4$, $\frac{10}{10} = 1$, $\frac{0}{20} = 0$. So there are 4 such integers.

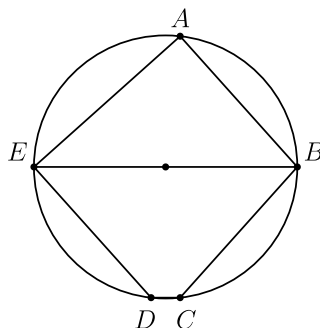
Watch out. $0 = 0^2$ is the square of an integer. Leaving out $n = 0$ gives 3, choice C.

Answer: (D) 4

7 Geometry

Source: 2011 AMC 10B #17

In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio 4 : 5. What is the degree measure of angle BCD ?



- (A) 120 (B) 125 (C) 130 (D) 135 (E) 140

Key idea. A diameter is a signal for a right angle: A lies on the circle, so $\angle EAB = 90^\circ$, and the ratio 4 : 5 then pins down both other angles of $\triangle ABE$. Parallel lines carry one of those angles down to E , and a quadrilateral inscribed in a circle has opposite angles adding to 180° .

Step 1. Since $\overline{EB}$ is a diameter, the angle it subtends at A is a right angle: $\angle EAB = 90^\circ$. So $\angle AEB + \angle ABE = 90^\circ$. Splitting 90° in the ratio 4 : 5 gives $\angle AEB = 40^\circ$ and $\angle ABE = 50^\circ$.

Step 2. $\overline{AB} \parallel \overline{ED}$, with transversal $\overline{EB}$, and A and D lie on opposite sides of $\overline{EB}$. So $\angle DEB$ and $\angle ABE$ are alternate interior angles:

$$\angle DEB = \angle ABE = 50^\circ.$$

Step 3. The points E, D, C, B lie on the circle in that order, so $EDCB$ is an inscribed quadrilateral, and its opposite angles at E and C are supplementary:

$$\angle BCD = 180^\circ - \angle DEB = 180^\circ - 50^\circ = 130^\circ.$$

(The fact $\overline{EB} \parallel \overline{DC}$ makes $EDCB$ an isosceles trapezoid, which agrees: $\angle CDE = \angle BCD = 130^\circ$.)

Watch out. The angle carried to E by the parallel lines is $\angle ABE = 50^\circ$, the larger one. Using $\angle AEB = 40^\circ$ instead (or reading the ratio backwards) gives $180^\circ - 40^\circ = 140^\circ$, choice E.

Answer: (C) 130

8 Algebra

Source: 2004 AMC 10A #18

A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

- (A) 1 (B) 4 (C) 36 (D) 49 (E) 81

Key idea. Name the one unknown, the common difference d , and use the key fact about three terms of a geometric progression: the middle term squared equals the product of the outer two. The words “smallest possible” warn that the equation will have more than one solution — expect a quadratic.

Step 1. The arithmetic progression is 9, $9 + d$, $9 + 2d$. After the additions, the three numbers are

$$9, \quad 11 + d, \quad 29 + 2d.$$

Step 2. These form a geometric progression, so

$$(11 + d)^2 = 9(29 + 2d).$$

Expanding: $121 + 22d + d^2 = 261 + 18d$, so $d^2 + 4d - 140 = 0$, which factors as $(d + 14)(d - 10) = 0$. Thus $d = 10$ or $d = -14$.

Step 3. Find each geometric progression.

- $d = 10$: the numbers are 9, 21, 49 (ratio $\frac{7}{3}$).
- $d = -14$: the numbers are 9, -3 , 1 (ratio $-\frac{1}{3}$).

Both are genuine geometric progressions, so the third term is 49 or 1, and the smallest is 1.

Watch out. It is easy to throw away $d = -14$ because it is negative, or because it makes the arithmetic progression 9, -5 , -19 look odd. Keeping only $d = 10$ gives 49, choice D. Nothing in the problem says the terms are positive.

Answer: (A) 1

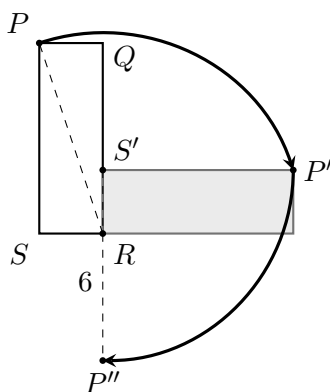
9 Geometry

Source: 2008 AMC 10A #19

Rectangle $PQRS$ lies in a plane with $PQ = RS = 2$ and $QR = SP = 6$. The rectangle is rotated 90° clockwise about R , then rotated 90° clockwise about the point that S moved to after the first rotation. What is the length of the path traveled by point P ?

- (A) $(2\sqrt{3} + \sqrt{5})\pi$ (B) 6π (C) $(3 + \sqrt{10})\pi$
 (D) $(\sqrt{3} + 2\sqrt{5})\pi$ (E) $2\sqrt{10}\pi$

Key idea. When a figure turns about a pivot, every point of it travels along an arc of a circle centered at the pivot, and its radius is that point's distance from the pivot. A 90° turn is a quarter circle. So the whole problem is: find P 's distance to each pivot.



Step 1. The first pivot is R . In the rectangle, P and R are opposite corners, so PR is a diagonal:

$$PR = \sqrt{PQ^2 + QR^2} = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}.$$

P moves along a quarter circle of radius $2\sqrt{10}$, of length $\frac{1}{4} \cdot 2\pi \cdot 2\sqrt{10} = \sqrt{10}\pi$.

Step 2. A rotation does not change distances, so after the first turn P' and S' are still $SP = 6$ apart, as sides of the moved rectangle. The second pivot is S' , so P' moves along a quarter circle of radius 6, of length $\frac{1}{4} \cdot 2\pi \cdot 6 = 3\pi$.

Step 3. The total path length is $3\pi + \sqrt{10}\pi = (3 + \sqrt{10})\pi$.

Watch out. S is a neighbour of P , not the corner opposite it. Using the diagonal $2\sqrt{10}$ as the radius for both turns gives $2 \cdot \sqrt{10}\pi = 2\sqrt{10}\pi$, choice E.

Answer: (C) $(3 + \sqrt{10})\pi$

10 Algebra

Source: 2014 AMC 10B #20

For how many integers x is the number $x^4 - 51x^2 + 50$ negative?

- (A) 8 (B) 10 (C) 12 (D) 14 (E) 16

Key idea. Only even powers of x appear, so treat x^2 as the variable: it is a quadratic in x^2 , and it factors. The sign of a product of two factors is easy to read off.

Step 1. Looking for two numbers with product 50 and sum -51 gives -1 and -50 :

$$x^4 - 51x^2 + 50 = (x^2 - 1)(x^2 - 50).$$

Step 2. A product is negative exactly when its two factors have opposite signs. Since $x^2 - 50$ is always smaller than $x^2 - 1$, the only way is

$$x^2 - 50 < 0 < x^2 - 1, \quad \text{that is,} \quad 1 < x^2 < 50.$$

Step 3. The perfect squares strictly between 1 and 50 are 4, 9, 16, 25, 36, 49, so $|x|$ is one of 2, 3, 4, 5, 6, 7. Each gives a positive and a negative x , for $2 \cdot 6 = 12$ integers.

Check. At $x = 7$: $(49 - 1)(49 - 50) = -48 < 0$. At $x = 8$: $(64 - 1)(64 - 50) > 0$. At $x = 1$: the value is 0.

Watch out. At $x = \pm 1$ the expression equals 0, which is not negative. Including $x = \pm 1$ (using $1 \leq x^2 < 50$) gives 14, choice D. Forgetting the negative values of x gives 6, which is not a choice.

Answer: (C) 12