
AMC 10

SET 2

HALF-LENGTH PRACTICE

SOLUTIONS

[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	C	2010 AMC 10B #10	Algebra
2	A	2012 AMC 10B #11	Counting & Probability
3	C	2009 AMC 10A #12	Geometry
4	D	2006 AMC 10B #14	Algebra
5	A	2014 AMC 10B #15	Geometry
6	C	2007 AMC 10B #16	Algebra
7	A	2016 AMC 10A #17	Counting & Probability
8	D	2019 AMC 10A #18	Number Theory
9	A	2014 AMC 10A #19	Geometry
10	C	2000 AMC 10 #20	Algebra

1 Algebra

Source: 2010 AMC 10B #10

Shelby drives her scooter at a speed of 30 miles per hour if it is not raining, and 20 miles per hour if it is raining. Today she drove in the sun in the morning and in the rain in the evening, for a total of 16 miles in 40 minutes. How many minutes did she drive in the rain?

- (A) 18 (B) 21 (C) 24 (D) 27 (E) 30

Key idea. Work in minutes. Call the time in the sun s minutes; then the rain time is $40 - s$, and the two pieces of distance must add up to 16 miles.

At 30 miles per hour Shelby covers $\frac{30}{60} = \frac{1}{2}$ mile each minute; at 20 miles per hour she covers $\frac{20}{60} = \frac{1}{3}$ mile each minute. So

$$\frac{1}{2}s + \frac{1}{3}(40 - s) = 16.$$

Multiply by 6: $3s + 2(40 - s) = 96$, so $s + 80 = 96$ and $s = 16$.

She drove 16 minutes in the sun, so she drove $40 - 16 = 24$ minutes in the rain.

Watch out. Solving for s gives the time in the *sun*, 16 minutes — not one of the choices, but a reminder to answer the question actually asked.

Another way. If all 40 minutes were in the rain she would cover $40 \cdot \frac{1}{3} = \frac{40}{3}$ miles, which is $16 - \frac{40}{3} = \frac{8}{3}$ miles short. Each minute switched to sun adds $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ mile, so she needs $\frac{8}{3} \div \frac{1}{6} = 16$ sunny minutes, leaving 24 in the rain.

Answer: (C) 24**2** Counting & Probability

Source: 2012 AMC 10B #11

A dessert chef prepares the dessert for every day of a week starting with Sunday. The dessert each day is either cake, pie, ice cream, or pudding. The same dessert may not be served two days in a row. There must be cake on Friday because of a birthday. How many different dessert menus for the week are possible?

- (A) 729 (B) 972 (C) 1024 (D) 2187 (E) 2304

Key idea. Start counting at the fixed day. Friday is forced to be cake, so count outward from Friday: each neighbouring day has exactly 3 choices, because it only has to differ from the day next to it that is already decided.

Step 1. Friday: cake — 1 way.

Step 2. Saturday must differ from Friday, so it is pie, ice cream, or pudding — 3 ways.

Step 3. Now go backwards. Thursday must differ from Friday's cake: 3 ways. Wednesday must differ from whatever Thursday is: 3 ways. In the same way Tuesday, Monday and Sunday each have 3 ways. That is 5 days with 3 choices each.

Step 4. Multiply: $1 \cdot 3 \cdot 3^5 = 3^6 = 729$.

Counting backwards is what makes this clean. Counting forwards from Sunday, the number of options for Thursday depends on whether Wednesday was cake, so the choices would not simply multiply.

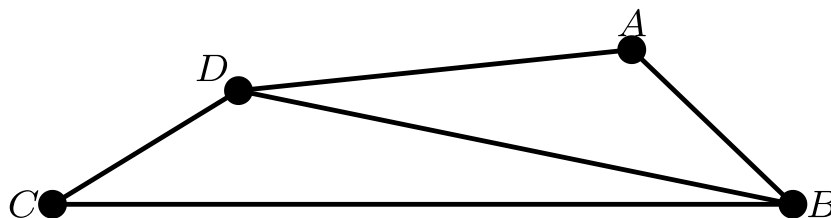
Watch out. Forgetting that Saturday also borders Friday and giving it all 4 desserts gives $3^5 \cdot 4 = 972$, choice (B). Counting forwards (4 for Sunday, 3 each for Monday–Thursday, 1 for Friday, 3 for Saturday) also gives 972 — but it is wrong, because if Thursday happens to be cake, Friday cannot be.

Answer: (A) 729

3 Geometry

Source: 2009 AMC 10A #12

In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD ?



- (A) 11 (B) 12 (C) 13 (D) 14 (E) 15

Key idea. The diagonal $\overline{BD}$ is a side of two triangles, $\triangle BCD$ and $\triangle ABD$. All other sides are known, so the triangle inequality in each triangle squeezes BD between two bounds — and the question says it is an integer, a strong hint that the bounds will leave only one value.

Step 1. In $\triangle BCD$ the sides are $BC = 17$, $CD = 5$ and BD . Any two sides of a triangle add to more than the third, so in particular $CD + BD > BC$, that is, $5 + BD > 17$. Hence $BD > 12$.

Step 2. In $\triangle ABD$ the sides are $AB = 5$, $DA = 9$ and BD . We need $5 + 9 > BD$, so $BD < 14$.

Step 3. The only integer with $12 < BD < 14$ is $BD = 13$. (It works: $13 < 5 + 17$ in $\triangle BCD$ and $13 > 9 - 5$ in $\triangle ABD$, so both triangles exist.)

Watch out. Writing $5 + BD \geq 17$ instead of a strict inequality allows $BD = 12$, choice (B). But then $5 + 12 = 17$ exactly, so C , D , B would lie on one straight line and there would be no triangle BCD .

Answer: (C) 13

4 Algebra

Source: 2006 AMC 10B #14

Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q ?

- (A) $\frac{5}{2}$ (B) $\frac{7}{2}$ (C) 4 (D) $\frac{9}{2}$ (E) 8

Key idea. We never need a , b or m themselves. In $x^2 - px + q = 0$ the constant term q is the *product* of the roots, and from $x^2 - mx + 2 = 0$ we know the product $ab = 2$ (Vieta's formulas). So expand the product of the new roots and write it in terms of ab .

Step 1. From $x^2 - mx + 2 = 0$, the product of the roots is $ab = 2$.

Step 2. The product of the roots of $x^2 - px + q = 0$ is q , so

$$q = \left(a + \frac{1}{b}\right) \left(b + \frac{1}{a}\right).$$

Step 3. Expand term by term:

$$q = ab + a \cdot \frac{1}{a} + \frac{1}{b} \cdot b + \frac{1}{ab} = ab + 1 + 1 + \frac{1}{ab}.$$

Step 4. Substitute $ab = 2$:

$$q = 2 + 2 + \frac{1}{2} = \frac{9}{2}.$$

Notice that m dropped out completely, which is why the problem did not need to tell us its value.

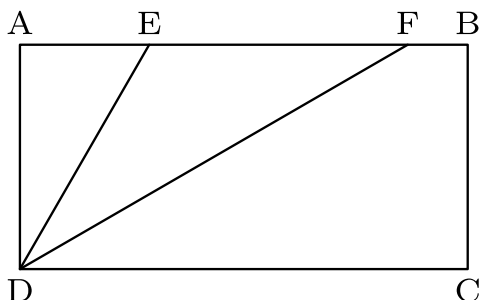
Watch out. Multiplying only “first times first” and “last times last” gives $ab + \frac{1}{ab} = \frac{5}{2}$, choice (A). The two cross terms $a \cdot \frac{1}{a}$ and $\frac{1}{b} \cdot b$ each equal 1 and must be included.

Answer: (D) $\frac{9}{2}$

5 Geometry

Source: 2014 AMC 10B #15

In rectangle $ABCD$, $DC = 2 \cdot CB$ and points E and F lie on $\overline{AB}$ so that $\overline{ED}$ and $\overline{FD}$ trisect $\angle ADC$ as shown. What is the ratio of the area of $\triangle DEF$ to the area of rectangle $ABCD$?



- (A) $\frac{\sqrt{3}}{6}$ (B) $\frac{\sqrt{6}}{8}$ (C) $\frac{3\sqrt{3}}{16}$ (D) $\frac{1}{3}$ (E) $\frac{\sqrt{2}}{4}$

Key idea. A trisected right angle means 30° angles, and 30° – 60° – 90° triangles have known side ratios. Triangle DEF has its base $\overline{EF}$ on $\overline{AB}$ and its height equal to AD , so we only need the lengths AE and AF .

Step 1. Let $AD = CB = 1$. Then $DC = AB = 2$ and the rectangle has area 2.

Step 2. $\angle ADC = 90^\circ$ is split into three 30° angles, so $\angle ADE = 30^\circ$ and $\angle ADF = 60^\circ$.

Step 3. Triangle DAE has a right angle at A and $\angle ADE = 30^\circ$. Its leg AE , opposite the 30° angle, is $\frac{1}{\sqrt{3}}$ times the other leg AD : $AE = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Triangle DAF has a right angle at A and $\angle ADF = 60^\circ$, so AF is $\sqrt{3}$ times AD : $AF = \sqrt{3}$. (This is less than $AB = 2$, so F really is on $\overline{AB}$.)

Step 4. $EF = AF - AE = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$. The height of $\triangle DEF$ from D to line AB is $AD = 1$, so

$$[DEF] = \frac{1}{2} \cdot \frac{2\sqrt{3}}{3} \cdot 1 = \frac{\sqrt{3}}{3}.$$

Step 5. The ratio is $\frac{\sqrt{3}/3}{2} = \frac{\sqrt{3}}{6}$.

Answer: (A) $\frac{\sqrt{3}}{6}$

6 Algebra

Source: 2007 AMC 10B #16

A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was 84. The juniors all received the same score, and the average score of the seniors was 83. What score did each of the juniors receive on the test?

- (A) 85 (B) 88 (C) 93 (D) 94 (E) 98

Key idea. Only percentages are given, so pick a convenient class size. With 10 students there is exactly 1 junior and 9 seniors, and an average turns into a total by multiplying by the number of students.

Step 1. Take a class of 10: 1 junior and 9 seniors. The class average is 84, so all ten scores add up to $10 \cdot 84 = 840$.

Step 2. The seniors average 83, so their nine scores add up to $9 \cdot 83 = 747$.

Step 3. The junior's score is what is left: $840 - 747 = 93$.

Any other class size gives the same answer, because all juniors have the same score: with $10n$ students, $10n \cdot 84 - 9n \cdot 83 = 93n$ points are shared equally by n juniors.

Watch out. Averaging the two group averages, $\frac{83+J}{2} = 84$, gives $J = 85$, choice (A). That would be right only if there were as many juniors as seniors; here the seniors outnumber the juniors 9 to 1, so they pull the class average much more.

Another way. Think of balancing around the class average 84. Each of the 9 seniors (on average) is 1 point below, for 9 points below in total. The single junior must make up those 9 points, so the junior scored $84 + 9 = 93$.

Answer: (C) 93

7 Counting & Probability

Source: 2016 AMC 10A #17

Let N be a positive multiple of 5. One red ball and N green balls are arranged in a line in random order. Let $P(N)$ be the probability that at least $\frac{3}{5}$ of the green balls are on the same side of the red ball. Observe that $P(5) = 1$ and that $P(N)$ approaches $\frac{4}{5}$ as N grows large. What is the sum of the digits of the least value of N such that $P(N) < \frac{321}{400}$?

- (A) 12 (B) 14 (C) 16 (D) 18 (E) 20

Key idea. The only thing that matters is *where the red ball lands*: if k green balls are to its left, then $N - k$ are to its right. In a random arrangement every value $k = 0, 1, \dots, N$ is equally likely, so $P(N)$ is just a count of good values of k out of $N + 1$. The given checks $P(5) = 1$ and the limit $\frac{4}{5}$ let you test your formula.

Step 1. Why are the $N + 1$ values of k equally likely? The red ball is equally likely to be in any of the $N + 1$ positions in the line, and if it is in position $k + 1$, exactly k green balls are to its left.

Step 2. The event fails only when *both* sides have fewer than $\frac{3}{5}N$ green balls: $k < \frac{3}{5}N$ and $N - k < \frac{3}{5}N$, i.e.

$$\frac{2}{5}N < k < \frac{3}{5}N.$$

Since N is a multiple of 5, both ends are integers, and the bad values are $k = \frac{2}{5}N + 1, \dots, \frac{3}{5}N - 1$. That is $\frac{1}{5}N - 1$ values.

Step 3. Hence

$$P(N) = 1 - \frac{\frac{N}{5} - 1}{N + 1} = \frac{\frac{4}{5}N + 2}{N + 1} = \frac{4N + 10}{5N + 5}.$$

Check: $P(5) = \frac{30}{30} = 1$, and for huge N this is close to $\frac{4N}{5N} = \frac{4}{5}$. Both match the statement.

Step 4. Solve $\frac{4N + 10}{5N + 5} < \frac{321}{400}$. Both denominators are positive, so cross-multiply:

$$400(4N + 10) < 321(5N + 5) \iff 1600N + 4000 < 1605N + 1605 \iff 2395 < 5N.$$

So $N > 479$. The least multiple of 5 above 479 is $N = 480$, with digit sum $4 + 8 + 0 = 12$.

(Indeed $P(475) = \frac{191}{238} \approx 0.80252$, just above $\frac{321}{400} = 0.8025$, while $P(480) = \frac{386}{481} \approx 0.80249$ is below it.)

Watch out. Stopping at $5N = 2395$ and answering $N = 479$ gives digit sum $4 + 7 + 9 = 20$, choice (E). But the inequality is strict, and 479 is not even a multiple of 5.

Answer: (A) 12

8 Number Theory

Source: 2019 AMC 10A #18

For some positive integer k , the repeating base- k representation of the (base-ten) fraction $\frac{7}{51}$ is

$$0.\overline{23}_k = 0.232323\dots_k.$$

What is k ?

- (A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Key idea. A repeating decimal is turned into a fraction by shifting it one full period and subtracting — and this works in any base. The period here is two digits, so multiply by k^2 .

Step 1. Let $x = 0.\overline{23}_k$. Multiplying by k^2 shifts the digits two places left:

$$k^2x = 23.\overline{23}_k = 23_k + x.$$

In base k , $23_k = 2k + 3$. Subtracting x from both sides gives $(k^2 - 1)x = 2k + 3$, so

$$x = \frac{2k + 3}{k^2 - 1}.$$

Step 2. Set this equal to $\frac{7}{51}$ and cross-multiply:

$$51(2k + 3) = 7(k^2 - 1) \iff 7k^2 - 102k - 160 = 0.$$

Step 3. Factor: $7k^2 - 102k - 160 = (k - 16)(7k + 10)$. (Check: $-112k + 10k = -102k$ and $-16 \cdot 10 = -160$.) Since k is a positive integer, $k = 16$.

Check: $\frac{2 \cdot 16 + 3}{16^2 - 1} = \frac{35}{255} = \frac{7}{51}$.

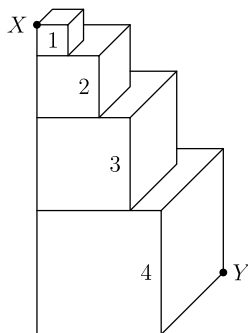
Another way. Use the choices. From $51(2k + 3) = 7(k^2 - 1)$, the prime 17 (a factor of 51) divides $7(k^2 - 1)$, so it divides $k^2 - 1 = (k - 1)(k + 1)$. Among 13–17, only $k = 16$ has $k - 1$ or $k + 1$ divisible by 17, and the check above confirms it.

Answer: (D) 16

9 Geometry

Source: 2014 AMC 10A #19

Four cubes with edge lengths 1, 2, 3, and 4 are stacked as shown. What is the length of the portion of $\overline{XY}$ contained in the cube with edge length 3?



- (A) $\frac{3\sqrt{33}}{5}$ (B) $2\sqrt{3}$ (C) $\frac{2\sqrt{33}}{3}$ (D) 4 (E) $3\sqrt{2}$

Key idea. Put the stack in coordinates. Along a straight segment every coordinate changes at a steady rate, so the fraction of $\overline{XY}$ inside the edge-3 cube equals the fraction of the total *height* it spans — provided the segment really stays inside that cube over those heights, which we must check.

Step 1. Put the common front-left vertical edge of the cubes on the z -axis, with the floor at $z = 0$. The cube of edge s occupies $0 \leq x \leq s$, $0 \leq y \leq s$, and the cubes are stacked at heights

$$\text{edge 4: } 0 \leq z \leq 4, \quad \text{edge 3: } 4 \leq z \leq 7, \quad \text{edge 2: } 7 \leq z \leq 9, \quad \text{edge 1: } 9 \leq z \leq 10.$$

Then $X = (0, 0, 10)$, the top front-left corner of the small cube, and $Y = (4, 4, 0)$, the bottom back-right corner of the big cube.

Step 2. $XY = \sqrt{4^2 + 4^2 + 10^2} = \sqrt{132} = 2\sqrt{33}$.

Step 3. Going from X to Y , the height falls from 10 to 0 at a steady rate. At height 7 we are $\frac{3}{10}$ of the way along, at the point $(1.2, 1.2, 7)$; at height 4 we are $\frac{6}{10}$ of the way, at $(2.4, 2.4, 4)$. Both points lie in the 3×3 square $0 \leq x, y \leq 3$, so the segment enters the cube through its top face, leaves through its bottom face, and stays inside in between.

Step 4. The portion inside the edge-3 cube is therefore $\frac{6}{10} - \frac{3}{10} = \frac{3}{10}$ of $\overline{XY}$:

$$\frac{3}{10} \cdot 2\sqrt{33} = \frac{3\sqrt{33}}{5}.$$

Watch out. Step 3 is not a formality. If the segment poked out through a side face of the cube, the portion inside would be shorter than $\frac{3}{10}$ of XY . Also, the piece inside is not a space diagonal of the cube: that would be $3\sqrt{3}$, which is not a choice.

Answer: (A) $\frac{3\sqrt{33}}{5}$

10 Algebra

Source: 2000 AMC 10 #20

Let A , M , and C be nonnegative integers such that $A + M + C = 10$. What is the maximum value of

$$A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A?$$

- (A) 49 (B) 59 (C) 69 (D) 79 (E) 89

Key idea. The expression contains the product of all three variables and every product of two — exactly the terms that appear when you expand $(A+1)(M+1)(C+1)$. Adding the missing pieces turns the problem into maximizing a product of three numbers with a fixed sum.

Step 1. Expand:

$$(A+1)(M+1)(C+1) = AMC + (AM + MC + CA) + (A + M + C) + 1.$$

Since $A + M + C = 10$,

$$AMC + AM + MC + CA = (A+1)(M+1)(C+1) - 11.$$

Step 2. Let $x = A+1$, $y = M+1$, $z = C+1$. These are positive integers with $x + y + z = 13$, and we must make xyz as large as possible.

Step 3: the best is at most 80. Suppose two of the numbers differ by 2 or more, say $x \geq y + 2$. Replace them by $x-1$ and $y+1$; the sum stays 13, and

$$(x-1)(y+1) = xy + (x-y-1) > xy,$$

because $x-y-1 \geq 1$. There are only finitely many triples, so a largest product exists, and by this argument it cannot have two numbers differing by 2 or more: all three must be within 1 of each other.

Three integers within 1 of each other summing to 13 must be 4, 4, 5, so the largest possible product is $4 \cdot 4 \cdot 5 = 80$.

Step 4: 80 is achieved. $x, y, z = 4, 4, 5$ means $A, M, C = 3, 3, 4$. Check directly: $3 \cdot 3 \cdot 4 + 3 \cdot 3 + 3 \cdot 4 + 4 \cdot 3 = 36 + 9 + 12 + 12 = 69$.

So the maximum is $80 - 11 = 69$.

Watch out. Forgetting the linear terms $A + M + C$ in the expansion, and writing the expression as $(A + 1)(M + 1)(C + 1) - 1$, gives $80 - 1 = 79$, choice (D).

Answer: (C) 69
