
AMC 10

SET 3

HALF-LENGTH PRACTICE

SOLUTIONS

[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	D	2012 AMC 10B #10	Number Theory
2	D	2008 AMC 10A #11	Algebra
3	C	2010 AMC 10A #12	Geometry
4	C	2015 AMC 10A #13	Counting & Probability
5	D	2015 AMC 10B #14	Algebra
6	A	2011 AMC 10B #16	Geometry
7	C	2011 AMC 10A #17	Algebra
8	B	2007 AMC 10B #18	Geometry
9	D	2001 AMC 10 #19	Counting & Probability
10	D	2007 AMC 10A #20	Algebra

1 Number Theory

Source: 2012 AMC 10B #10

How many ordered pairs of positive integers (M, N) satisfy the equation

$$\frac{M}{6} = \frac{6}{N}?$$

- (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Key idea. Cross-multiply to turn the equation into a product: $MN = 36$. Then every pair is just a way of splitting 36 into two factors.

Cross-multiplying $\frac{M}{6} = \frac{6}{N}$ gives $MN = 36$. Once M is chosen as a positive divisor of 36, the value $N = \frac{36}{M}$ is forced, and it is a positive integer. So we only need to count the divisors of 36:

1, 2, 3, 4, 6, 9, 12, 18, 36.

That is 9 choices of M , hence 9 ordered pairs, from $(1, 36)$ to $(36, 1)$.

As a check, $36 = 2^2 \cdot 3^2$, so it has $(2 + 1)(2 + 1) = 9$ divisors.

Watch out. Listing the factor pairs $(1, 36), (2, 18), (3, 12), (4, 9)$ and doubling them for order gives 8, choice (C). The pair $(6, 6)$ is its own reverse and must be counted once more.

Answer: (D) 9**2** Algebra

Source: 2008 AMC 10A #11

While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing toward the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?

- (A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Key idea. First find how long the trip takes; then the boat may gain at most 30 gallons in that time. The leak is in gallons per *minute* but the speed is in miles per *hour*, so convert the trip time to minutes before anything else.

Step 1. At 4 miles per hour, 1 mile takes $\frac{1}{4}$ hour, which is $\frac{1}{4} \cdot 60 = 15$ minutes.

Step 2. In those 15 minutes the leak lets in $15 \cdot 10 = 150$ gallons.

Step 3. The boat survives as long as it holds no more than 30 gallons, so LeRoy must remove at least

$150 - 30 = 120$ gallons in 15 minutes. That is

$$\frac{120}{15} = 8 \text{ gallons per minute.}$$

Check: bailing at 8, the water rises at a net $10 - 8 = 2$ gallons per minute, and $2 \cdot 15 = 30$ gallons at the shore — exactly the limit, which is not *more* than 30. Any slower rate, and the net gain $10 - r$ times 15 exceeds 30.

Watch out. Forgetting that the boat can hold 30 gallons and bailing *all* the incoming water gives 10, choice (E). That rate works, but it is not the slowest one.

Answer: (D) 8

3 Geometry

Source: 2010 AMC 10A #12

Logan is constructing a scaled model of his town. The city's water tower stands 40 meters high, and the top portion is a sphere that holds 100,000 liters of water. Logan's miniature water tower holds 0.1 liters. How tall, in meters, should Logan make his tower?

- (A) 0.04 (B) $\frac{0.4}{\pi}$ (C) 0.4 (D) $\frac{4}{\pi}$ (E) 4

Key idea. In a scale model every length shrinks by the same factor k , so every volume shrinks by k^3 . The volumes are given and a height is asked for, which tells you to take a cube root.

Step 1. Compare the volumes:

$$\frac{100,000}{0.1} = 1,000,000.$$

The real sphere holds a million times as much water as the model sphere.

Step 2. Volume scales as the cube of length. If the real tower is k times as tall as the model, then $k^3 = 1,000,000 = 100^3$, so $k = 100$.

Step 3. The model's height is

$$\frac{40}{100} = 0.4 \text{ meters.}$$

Notice that we never needed the formula $\frac{4}{3}\pi r^3$ for the volume of a sphere: the π is the same in both towers and cancels in the ratio, so the choices containing π are a distraction.

Watch out. Treating the volume ratio like an area ratio and taking a square root gives $k = 1000$ and a height of $\frac{40}{1000} = 0.04$, choice (A). Areas scale by k^2 ; volumes scale by k^3 .

Answer: (C) 0.4

4 Counting & Probability

Source: 2015 AMC 10A #13

Claudia has 12 coins, each of which is a 5-cent coin or a 10-cent coin. There are exactly 17 different values that can be obtained as combinations of one or more of her coins. How many 10-cent coins does Claudia have?

- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Key idea. Every value is a multiple of 5 cents, so ask *which* multiples of 5 can be made. With at least one 5-cent coin you can fill every gap, so the count of values is simply the total value divided by 5.

Step 1. Let Claudia have x five-cent coins and $12 - x$ ten-cent coins. If $x = 0$, the possible values are 10, 20, $\dots$, 120 — only 12 values, not 17. So $x \geq 1$.

Step 2. With $x \geq 1$, every multiple of 5 from 5 cents up to her total is possible. To make $5k$ cents (up to her total), use as many dimes as possible and nickels for the rest. If the dimes do not run out, the rest is at most one nickel. If they do run out, the rest is the amount above all her dimes, which is at most her x nickels.

Step 3. Her total is

$$5x + 10(12 - x) = 120 - 5x \text{ cents,}$$

so the values are 5, 10, $\dots$, $120 - 5x$, which is $\frac{120 - 5x}{5} = 24 - x$ values.

Step 4. Setting $24 - x = 17$ gives $x = 7$ five-cent coins, so she has $12 - 7 = 5$ ten-cent coins.

Check: 7 nickels and 5 dimes total $35 + 50 = 85$ cents, and 5, 10, $\dots$, 85 is 17 values.

Watch out. Solving for x and stopping there gives 7, choice (E) — but x counts the 5-cent coins. The question asks for the 10-cent coins.

Answer: (C) 5

5 Algebra

Source: 2015 AMC 10B #14

Let a , b , and c be three distinct one-digit numbers. What is the maximum value of the sum of the roots of the equation

$$(x - a)(x - b) + (x - b)(x - c) = 0?$$

- (A) 15 (B) 15.5 (C) 16 (D) 16.5 (E) 17

Key idea. Both products contain the factor $(x - b)$ — pull it out and the roots appear directly. Then the sum of the roots is a weighted sum of a, b, c , and the heaviest weight gets the biggest digit.

Step 1. Factor out $(x - b)$:

$$(x - b)[(x - a) + (x - c)] = (x - b)(2x - a - c) = 0.$$

The roots are $x = b$ and $x = \frac{a+c}{2}$.

Step 2. Their sum is

$$S = b + \frac{a + c}{2} = \frac{a + 2b + c}{2}.$$

The digit b counts twice, while a and c count once each.

Step 3. Upper bound. Write $a + 2b + c = (a + b + c) + b$. Three distinct digits add to at most $9 + 8 + 7 = 24$, and $b \leq 9$, so $a + 2b + c \leq 33$ and $S \leq 16.5$.

Step 4. It is achieved. Take $b = 9$, $a = 7$, $c = 8$. The equation becomes $(x - 9)(2x - 15) = 0$, with roots 9 and 7.5, whose sum is 16.5.

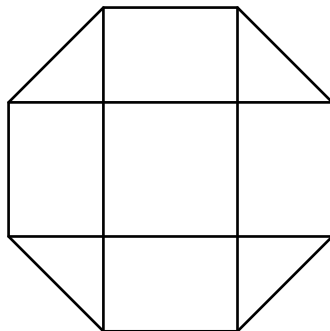
Watch out. Putting the 9 on a or c instead of on b — for example $a = 9$, $b = 8$, $c = 7$ — gives $8 + \frac{9+7}{2} = 16$, choice (C). The factored form shows that b is worth twice as much.

Answer: (D) 16.5

6 Geometry

Source: 2011 AMC 10B #16

A dart board is a regular octagon divided into regions as shown. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands within the center square?



- (A) $\frac{\sqrt{2}-1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{2-\sqrt{2}}{2}$ (D) $\frac{\sqrt{2}}{4}$ (E) $2-\sqrt{2}$

Key idea. “Equally likely to land anywhere” means the probability is an area ratio: center square over whole octagon. Since only a ratio is asked, choose a convenient size — let the octagon’s side be 1.

Step 1. The two vertical lines start at the ends of the octagon’s top side, so the center square is exactly as wide as that side. With octagon side 1, the center square has side 1 and area 1.

Step 2. Each corner piece is an isosceles right triangle whose hypotenuse is a side of the octagon, length 1. Its legs are therefore $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, and its area is $\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{4}$. The four corners together have area 1.

Step 3. Each of the four rectangles has one side 1 (shared with the center square) and the other side equal to a triangle leg, $\frac{\sqrt{2}}{2}$. Together they have area $4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}$.

Step 4. The octagon’s area is $1 + 1 + 2\sqrt{2} = 2 + 2\sqrt{2}$, so the probability is

$$\frac{1}{2 + 2\sqrt{2}} = \frac{1}{2(\sqrt{2} + 1)} \cdot \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = \frac{\sqrt{2} - 1}{2(2 - 1)} = \frac{\sqrt{2} - 1}{2}.$$

This is about 0.207.

Watch out. Dividing by the area of the bounding square, $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$, forgets to remove the four cut-off corners and gives $3 - 2\sqrt{2} \approx 0.17$, which is not a choice.

Answer: (A) $\frac{\sqrt{2}-1}{2}$

7 Algebra

Source: 2011 AMC 10A #17

In the eight term sequence A, B, C, D, E, F, G, H , the value of C is 5 and the sum of any three consecutive terms is 30. What is $A + H$?

- (A) 17 (B) 18 (C) 25 (D) 26 (E) 43

Key idea. Compare two overlapping sums: $A + B + C = 30$ and $B + C + D = 30$ share $B + C$, so $D = A$. The same comparison all along shows the sequence repeats every three terms — then just see where H falls.

Step 1. Subtract consecutive sums. From $(B + C + D) - (A + B + C) = 30 - 30$ we get $D = A$. In the same way $E = B$, $F = C$, $G = D$, and $H = E$.

Step 2. So the sequence repeats with period 3:

$$A, B, 5, A, B, 5, A, B,$$

using $C = 5$ and $F = C = 5$. In particular $H = B$.

Step 3. From $A + B + C = 30$ we get $A + B = 30 - 5 = 25$, so

$$A + H = A + B = 25.$$

Note that A itself is not determined — any A works, with $B = 25 - A$ — but the sum $A + H$ always equals 25.

Another way. Combine five of the triple sums so that almost everything cancels:

$$(A + B + C) + (C + D + E) + (F + G + H) - (B + C + D) - (E + F + G) = A + C + H.$$

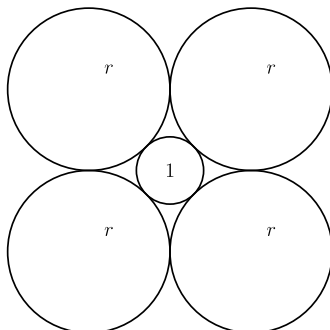
The left side is $30 + 30 + 30 - 30 - 30 = 30$, so $A + C + H = 30$ and $A + H = 30 - 5 = 25$.

Answer: (C) 25

8 Geometry

Source: 2007 AMC 10B #18

A circle of radius 1 is surrounded by 4 circles of radius r as shown. What is r ?



- (A) $\sqrt{2}$ (B) $1 + \sqrt{2}$ (C) $\sqrt{6}$ (D) 3 (E) $2 + \sqrt{2}$

Key idea. For tangent circles, the distance between centers is the sum of the radii. Joining the four outer centers makes a square, and the small circle's center sits at the middle of that square — so compute one distance in two ways.

Step 1. Adjacent outer circles are tangent, so the distance between their centers is $r + r = 2r$. The four outer centers form a square of side $2r$.

Step 2. The small circle touches all four outer circles, so its center is the same distance from all four corners — it is the center of the square. The distance from the center of a square to a corner is half the diagonal:

$$\frac{2r\sqrt{2}}{2} = r\sqrt{2}.$$

Step 3. The small circle is tangent to each outer circle, so the same distance is also $1 + r$. Setting them equal,

$$r\sqrt{2} = r + 1 \implies r(\sqrt{2} - 1) = 1 \implies r = \frac{1}{\sqrt{2} - 1}.$$

Step 4. Rationalize: $\frac{1}{\sqrt{2} - 1} \cdot \frac{\sqrt{2} + 1}{\sqrt{2} + 1} = \frac{\sqrt{2} + 1}{2 - 1} = 1 + \sqrt{2}$.

Check: with $r = 1 + \sqrt{2}$, $r\sqrt{2} = \sqrt{2} + 2$ and $r + 1 = 2 + \sqrt{2}$. They agree.

Answer: (B) $1 + \sqrt{2}$

9 Counting & Probability

Source: 2001 AMC 10 #19

Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

- (A) 6 (B) 9 (C) 12 (D) 15 (E) 18

Key idea. A selection is decided only by *how many* of each type Pat takes — order does not matter and the supply never runs out. So count solutions of $g + c + p = 4$ in nonnegative integers; this is exactly what “stars and bars” counts.

Step 1. Let g, c, p be the numbers of glazed, chocolate and powdered donuts. Each selection is one solution of

$$g + c + p = 4, \quad g, c, p \geq 0.$$

Step 2. Picture the four donuts as stars, and use two bars to split them into three groups (glazed | chocolate | powdered). For example $\star\star || \star\star$ means 2 glazed, 0 chocolate, 2 powdered.

Step 3. Every arrangement of 4 stars and 2 bars in a row of 6 places gives exactly one selection, and vice versa. Choosing the 2 places for the bars gives

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 15.$$

Another way. Count by the shape of the selection. All four alike (4+0+0): 3 ways. Three and one (3+1+0): $3 \cdot 2 = 6$ ways. Two and two (2+2+0): 3 ways. Two, one, one (2+1+1): 3 ways. Total $3 + 6 + 3 + 3 = 15$.

Watch out. In the case count it is easy to forget a pattern. Missing the 4+0+0 selections or the 2+2+0 selections drops 3 and gives 12, choice (C).

Answer: (D) 15

10 Algebra

Source: 2007 AMC 10A #20

Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

- (A) 164 (B) 172 (C) 192 (D) 194 (E) 212

Key idea. Don't solve for a . Square the given equation instead: because $a \cdot a^{-1} = 1$, the cross term is just the number 2, and you climb from power 1 to power 2 to power 4.

Step 1. Square $a + a^{-1} = 4$:

$$a^2 + 2 \cdot a \cdot a^{-1} + a^{-2} = 16, \quad \text{so} \quad a^2 + 2 + a^{-2} = 16,$$

and $a^2 + a^{-2} = 14$.

Step 2. Square again:

$$a^4 + 2 \cdot a^2 \cdot a^{-2} + a^{-4} = 196, \quad \text{so} \quad a^4 + 2 + a^{-4} = 196,$$

and $a^4 + a^{-4} = 194$.

Watch out. Squaring as if $(x + y)^2 = x^2 + y^2$ drops the middle term 2. Doing that in the last step gives 196, and in both steps gives 256 — neither is a choice, which should send you back to check the cross terms.

Another way. Solve directly. Multiplying by a gives $a^2 - 4a + 1 = 0$, so $a = 2 \pm \sqrt{3}$. Take $a = 2 + \sqrt{3}$ (the other root is its reciprocal, so it gives the same answer). Then $a^2 = 7 + 4\sqrt{3}$ and $a^4 = (7 + 4\sqrt{3})^2 = 49 + 56\sqrt{3} + 48 = 97 + 56\sqrt{3}$. Since $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$, we have $a^{-4} = 97 - 56\sqrt{3}$, and the sum is 194.

Answer: (D) 194