AMC 10

SET 4

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**E**	2017 AMC 10B #10	Geometry
2	**D**	2011 AMC 10B #11	Counting & Probability
3	**C**	2003 AMC 10B #12	Algebra
4	**D**	2007 AMC 10B #13	Geometry
5	**B**	2006 AMC 10A #14	Algebra
6	**C**	2016 AMC 10B #15	Counting & Probability
7	**D**	2016 AMC 10A #16	Geometry
8	**D**	2013 AMC 10B #18	Number Theory
9	**B**	2005 AMC 10B #19	Algebra
10	**B**	2018 AMC 10A #20	Counting & Probability

1 Geometry

Source: 2017 AMC 10B #10

> The lines with equations $ax - 2y = c$ and $2x + by = -c$ are perpendicular and intersect at $(1, -5)$. What is c?
>
> **(A)** -13 **(B)** -8 **(C)** 2 **(D)** 8 **(E)** 13

Key idea. There are three unknowns, a, b, c, and the statement gives exactly three facts: one about the slopes and two from the shared point. Turn each into an equation.

Step 1. Solve each equation for y: the first is $y = \frac{a}{2}x - \frac{c}{2}$, with slope $\frac{a}{2}$; the second is $y = -\frac{2}{b}x - \frac{c}{b}$, with slope $-\frac{2}{b}$. Perpendicular slopes multiply to -1:

$$\frac{a}{2} \cdot \left(-\frac{2}{b}\right) = -1 \quad \Longrightarrow \quad a = b.$$

Step 2. The point $(1, -5)$ lies on both lines. In the first, $a + 10 = c$. In the second, $2 - 5b = -c$, and since $b = a$ this is $2 - 5a = -c$.

Step 3. Add the two equations: $(a + 10) + (2 - 5a) = c - c = 0$, so $12 - 4a = 0$ and $a = 3$. Then $c = a + 10 = 13$.

Check: the lines $3x - 2y = 13$ and $2x + 3y = -13$ have slopes $\frac{3}{2}$ and $-\frac{2}{3}$, and $(1, -5)$ satisfies both.

Watch out. Dropping the minus sign on the second slope (taking it as $\frac{2}{b}$) gives $a = -b$, and the same steps then lead to $c = 8$, choice (D).

Answer: (E) 13

2 Counting & Probability

Source: 2011 AMC 10B #11

> There are 52 people in a room. What is the largest value of n such that the statement "At least n people in this room have birthdays falling in the same month" is always true?
>
> **(A)** 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 12

Key idea. "Always true" means we must look at the *worst* possible room — the one that spreads birthdays as evenly as it can. Sorting 52 people into 12 months is a pigeonhole question, and the even split tells us everything.

Step 1: some month must have at least 5. Suppose every month had at most 4 birthdays. Then the room would hold at most $12 \times 4 = 48$ people. But there are 52, so at least one month has 5 or more. Hence the statement is always true for $n = 5$ (and so also for any smaller n).

Step 2: 6 can fail. We need one room where no month has 6 birthdays. Put 5 people in each of 4 months and 4 people in each of the other 8 months:

$$4 \times 5 + 8 \times 4 = 20 + 32 = 52.$$

In this room the most crowded month has only 5 birthdays, so "at least 6 in the same month" is false. The same room defeats every $n \geq 6$, including 12.

Both halves together: 5 always works and 6 does not, so the largest n is 5.

Watch out. The average is $\frac{52}{12} = 4\frac{1}{3}$, and rounding this *down* gives 4, choice (C). But since the average exceeds 4, some month must be *above* 4 — round up, not down.

Answer: (D) 5

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3 Algebra

Source: 2003 AMC 10B #12

Al, Betty, and Clare split \$1000 among them to be invested in different ways. Each begins with a different amount. At the end of one year they have a total of \$1500. Betty and Clare have both doubled their money, whereas Al has managed to lose \$100. What was Al's original portion?

(A) \$250 **(B)** \$350 **(C)** \$400 **(D)** \$450 **(E)** \$500

Key idea. Betty and Clare are treated identically — both double — so we never need their separate amounts, only their *combined* amount. That leaves one unknown besides Al's, and two totals to pin them down.

Step 1. Let Al start with a dollars. Then Betty and Clare together start with $1000 - a$ dollars.

Step 2. After one year, Al has $a-100$. Betty and Clare each doubled, so together they have $2(1000-a)$. The new total is 1500:

$$(a - 100) + 2(1000 - a) = 1500.$$

Step 3. Expand: $a - 100 + 2000 - 2a = 1500$, so $1900 - a = 1500$ and $a = 400$.

Check: Al starts with \$400 and ends with \$300; Betty and Clare start with \$600 together and end with \$1200. The total is $300 + 1200 = \$1500$, as required. The sentence "Each begins with a different amount" plays no role: Al's share is \$400 however Betty and Clare divide their \$600 (for example \$250 and \$350).

Another way. Think in terms of the change. The group gained $1500 - 1000 = \$500$. Al's change was $-\$100$, so Betty and Clare gained \$600. Doubling means each gained exactly what she started with, so they started with \$600 together, and Al started with $1000 - 600 = \$400$.

Answer: (C) \$400

4 Geometry

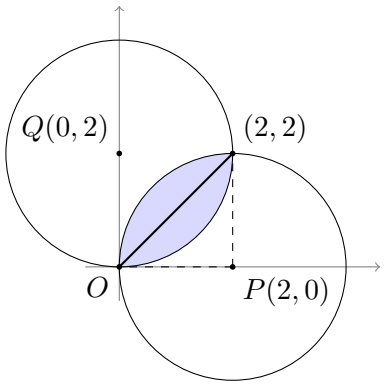
Source: 2007 AMC 10B #13

Two circles of radius 2 are centered at $(2,0)$ and at $(0,2)$. What is the area of the intersection of the interiors of the two circles?

(A) $\pi-2$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi\sqrt{3}}{3}$ **(D)** $2(\pi-2)$ **(E)** π

Key idea. A lens-shaped overlap is two circular *segments* glued along their common chord. A segment is "sector minus triangle", and here the sector is a clean quarter circle because the radii to the chord's ends are perpendicular.

Step 1: find the chord. Both circles pass through $(0,0)$ (its distance from $(2,0)$ and from $(0,2)$ is 2) and through $(2,2)$ (same check). So the overlap is cut in half by the segment from $(0,0)$ to $(2,2)$, and by symmetry across the line $y=x$ the two halves have equal area.



Step 2: one half. The upper-left half is bounded by the arc of the circle centered at $P(2,0)$. The radii from P go to $O(0,0)$ (pointing left) and to $(2,2)$ (pointing up), so they meet at a right angle. The quarter circle between them has area $\frac{1}{4}\pi(2)^2=\pi$. Remove the right triangle with legs 2 and 2, area $\frac{1}{2}\cdot 2\cdot 2=2$. The half-lens has area $\pi-2$.

Step 3: double it. The whole intersection is $2(\pi-2)$, about 2.28.

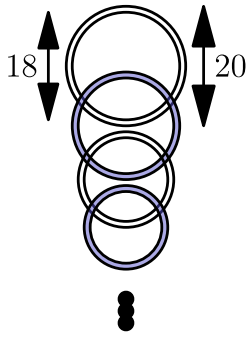
Watch out. $\pi-2$ is only *one* of the two segments. Stopping there gives choice (A).

Answer: (D) $2(\pi-2)$

5 Algebra

Source: 2006 AMC 10A #14

A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the other rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?



(A) 171 **(B)** 173 **(C)** 182 **(D)** 188 **(E)** 210

Key idea. Adding the outside diameters double-counts the overlaps. Look instead at the *holes*: the figure's "18" shows that each ring's hole begins exactly where the hole above it ends, so the holes stack end to end with no gaps and no overlaps.

Step 1: why the holes line up. Measure downward from the top of the top ring. Its 1 cm top bar fills 0 to 1, its hole runs from 1 to $1 + 18 = 19$, and its bottom bar fills 19 to 20. The second ring hooks over that bottom bar: the second ring's top bar rests on it from 18 to 19 (matching the 18 in the figure), and its hole starts at 19 — right where the first hole stopped. The same thing happens at every link.

Step 2: add the holes. A ring with outside diameter d and thickness 1 has a hole of $d - 2$. The outside diameters are $20, 19, \ldots, 3$ (that is 18 rings), so the holes are $18, 17, \ldots, 1$, with total

$$1 + 2 + \cdots + 18 = \frac{18 \cdot 19}{2} = 171.$$

Step 3: add the two ends. Above the first hole sits the top ring's 1 cm top bar, and below the last hole sits the bottom ring's 1 cm bottom bar. The total distance is $1 + 171 + 1 = 173$ cm.

Watch out. Adding only the holes gives 171, choice (A) — the 1 cm at the very top and at the very bottom are easy to forget.

Another way. Each ring after the first adds its outside diameter minus 2 (its top bar is 2 cm above the bottom of the ring before it), giving $20 + (17 + 16 + \cdots + 1) = 20 + 153 = 173$.

Answer: (B) 173

6 Counting & Probability

Source: 2016 AMC 10B #15

All the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ are written in a 3×3 array of squares, one number in each square, in such a way that if two numbers are consecutive then they occupy squares that share an edge. The numbers in the four corners add up to 18. What is the number in the center?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Key idea. Writing $1, 2, \ldots, 9$ so that each number is next to the one before is a path that steps from square to square. Paths on a grid are controlled by a *checkerboard coloring*: every step changes color. The statement singles out corners and center — exactly one color class of the checkerboard.

Step 1: color the board. Shade the four corners and the center; leave the four edge squares white. Any two squares that share an edge have different colors. There are 5 shaded squares and 4 white ones.

Step 2: odd numbers share a color. Since k and $k+1$ are on adjacent squares, the colors alternate along $1, 2, 3, \ldots, 9$. So $1, 3, 5, 7, 9$ are all one color and $2, 4, 6, 8$ are all the other. Five odd numbers cannot fit into the 4 white squares, so the odd numbers fill the 5 shaded squares: the corners and the center.

Step 3: subtract. Corners plus center $= 1 + 3 + 5 + 7 + 9 = 25$. The corners add to 18, so the center is $25 - 18 = 7$.

Such an array does exist, for example

$$\begin{array}{ccc} 1 & 2 & 3 \\ 8 & 7 & 4 \\ 9 & 6 & 5 \end{array}$$

with corners $1 + 3 + 9 + 5 = 18$ and 7 in the middle.

Watch out. Guessing that the middle number 5 sits in the middle square gives (A). But then the corners would add to $25 - 5 = 20$, not 18.

Answer: (C) 7

7 Geometry

Source: 2016 AMC 10A #16

A triangle with vertices $A(0,2)$, $B(-3,2)$, and $C(-3,0)$ is reflected about the x-axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by 90° to produce $\triangle A''B''C''$. Which of the following transformations will return $\triangle A''B''C''$ to $\triangle ABC$?

(A) counterclockwise rotation about the origin by 90°
(B) clockwise rotation about the origin by 90°
(C) reflection about the x-axis
(D) reflection about the line $y = x$
(E) reflection about the y-axis

Key idea. Two transformations in a row are hard to picture, but easy to compute: follow a general point (x, y) through both steps and see what single rule comes out. Then ask which choice undoes that rule.

Step 1: the two rules. Reflecting about the x-axis keeps x and negates y: $(x, y) \mapsto (x, -y)$. Rotating 90° counterclockwise about the origin sends $(p, q) \mapsto (-q, p)$; for instance $(1, 0)$, on the positive x-axis, goes to $(0, 1)$, on the positive y-axis.

Step 2: combine them. Start with (x, y). After the reflection it is $(x, -y)$. Rotating, with $p = x$ and $q = -y$, gives $(-(-y), x) = (y, x)$. So the whole process simply swaps the coordinates — it is the reflection about the line $y = x$.

Step 3: undo it. Swapping the coordinates twice gets you back where you started, so the reflection about $y = x$ is its own undo. Check with the vertices: $A'' = (2, 0)$, $B'' = (2, -3)$, $C'' = (0, -3)$, and swapping gives $(0, 2)$, $(-3, 2)$, $(-3, 0)$ — exactly A, B, C.

Watch out. A clockwise rotation (B) only undoes the *last* step. It takes $A'' = (2, 0)$ to $(0, -2) = A'$, so it returns the triangle to $\triangle A'B'C'$, not to $\triangle ABC$.

Answer: (D) reflection about $y = x$

8 Number Theory

Source: 2013 AMC 10B #18

> The number 2013 has the property that its units digit is the sum of its other digits, that is $2+0+1=3$. How many integers less than 2013 but greater than 1000 have this property?
>
> **(A)** 33 **(B)** 34 **(C)** 45 **(D)** 46 **(E)** 58

Key idea. Once the first three digits are chosen, the units digit is *forced* — it must equal their sum — and it exists exactly when that sum is at most 9. So we are really counting choices of the first three digits, and the thousands digit splits the range into two very unequal cases.

Step 1: thousands digit 1. Write the number as $1\,x\,y\,z$ with digits x, y, z. We need $z = 1+x+y$, which is a digit exactly when $1+x+y \le 9$, that is $x+y \le 8$. Every such number lies between 1001 and 1999, so all of them are in range.

Step 2: count the pairs. If $x+y=s$, then x can be any of $0, 1, \ldots, s$ (and $y = s-x$), which is $s+1$ choices. Summing over $s = 0, 1, \ldots, 8$:

$$1+2+3+\cdots+9 = \frac{9\cdot 10}{2} = 45.$$

Step 3: thousands digit 2. The number must be below 2013, so it is one of $2000, \ldots, 2012$, with hundreds digit 0 and tens digit 0 or 1. Tens digit 0: the units digit must be $2+0+0=2$, giving 2002. Tens digit 1: the units digit must be 3, giving 2013 itself, which is excluded.

Step 4. The total is $45+1=46$.

Watch out. Stopping after the numbers from 1001 to 1999 gives 45, choice (C). The single number 2002 is easy to miss.

Answer: (D) 46

9 Algebra

Source: 2005 AMC 10B #19

> On a certain math exam, 10% of the students got 70 points, 25% got 80 points, 20% got 85 points, 15% got 90 points, and the rest got 95 points. What is the difference between the mean and the median score on this exam?
>
> **(A)** 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 5

Key idea. Only percentages are given, so pretend the class has exactly 100 students: then every percentage is a head count, the mean is a weighted average, and the median is the score of the students in the middle of the line.

Step 1: the missing group. The students who got 95 make up $100-10-25-20-15=30$ percent, so 30 of our 100 students.

Step 2: the median. Line the 100 scores up from lowest to highest. Students 1–10 scored 70, students 11–35 scored 80, and students 36–55 scored 85. The median is the average of the 50th and 51st scores; both lie in the block 36–55, so the median is 85.

Step 3: the mean. The total of all 100 scores is

$$10(70) + 25(80) + 20(85) + 15(90) + 30(95) = 700 + 2000 + 1700 + 1350 + 2850 = 8600,$$

so the mean is $\frac{8600}{100} = 86$.

Step 4. The difference is $86 - 85 = 1$.

Another way. Measure every score from the median 85. The deviations are -15, -5, 0, $+5$, $+10$, so the mean is

$$85 + 0.10(-15) + 0.25(-5) + 0.20(0) + 0.15(5) + 0.30(10) = 85 - 1.5 - 1.25 + 0.75 + 3 = 86.$$

This shows the difference directly: the weighted deviations add up to 1.

Answer: (B) 1

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10 Counting & Probability

Source: 2018 AMC 10A #20

A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares.

A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides.

What is the total number of possible symmetric scanning codes?

(A) 510 **(B)** 1022 **(C)** 8190 **(D)** 8192 **(E)** 65,534

Key idea. Every symmetry condition says "these squares must have the same color." So the squares fall into groups that are forced to match, and a symmetric code is just a free choice of color for each group. The whole problem is to count the groups.

Step 1: coordinates. Number the rows and columns from -3 to 3, with the center square at $(0, 0)$. The 90° rotation sends (x, y) to $(-y, x)$; the reflections across the middle row and column change the sign of one coordinate; the reflections across the diagonals swap the two coordinates (possibly with both signs changed). Combining these, a square (x, y) can be carried to every square obtained by changing signs and/or swapping coordinates, and to no others.

Step 2: the groups. So two squares must match exactly when they have the same pair of distances $\{|x|, |y|\}$ from the center lines. Listing the pairs as (u, v) with $0 \le u \le v \le 3$ gives $4 + 3 + 2 + 1 = 10$ groups. In the grid below, squares with the same label form one group; the shaded squares show one representative of each.

10	9	8	7	8	9	10
9	6	5	4	5	6	9
8	5	3	2	3	5	8
7	4	2	1	2	4	7
8	5	3	2	3	5	8
9	6	5	4	5	6	9
10	9	8	7	8	9	10

Step 3: count. Every symmetry maps each group onto itself, so *any* choice of colors for the 10 groups gives a symmetric code. That is $2^{10} = 1024$ codes. Two of them are all black and all white, which the rules forbid, leaving $1024 - 2 = 1022$.

Watch out. Using only the rotations gives $(49 - 1)/4 + 1 = 13$ groups (each group of four squares, plus the center), and $2^{13} - 2 = 8190$, choice (C). The reflections merge more squares and cut the count to 10 groups.

Answer: (B) 1022