
AMC 10

SET 5

HALF-LENGTH PRACTICE

SOLUTIONS

[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	A	2008 AMC 10B #10	Geometry
2	E	2018 AMC 10A #11	Counting & Probability
3	C	2000 AMC 10 #12	Algebra
4	C	2018 AMC 10B #13	Number Theory
5	D	2017 AMC 10A #14	Algebra
6	B	2015 AMC 10B #15	Number Theory
7	B	2011 AMC 10A #16	Algebra
8	C	2009 AMC 10B #17	Geometry
9	D	2017 AMC 10B #18	Counting & Probability
10	A	2005 AMC 10A #20	Geometry

1 Geometry

Source: 2008 AMC 10B #10

Points A and B are on a circle of radius 5 and $AB = 6$. Point C is the midpoint of the minor arc AB . What is the length of the line segment AC ?

- (A) $\sqrt{10}$ (B) $\frac{7}{2}$ (C) $\sqrt{14}$ (D) $\sqrt{15}$ (E) 4

Key idea. Drop the perpendicular from the center O to the chord $\overline{AB}$. It hits the chord at its midpoint D and, extended, passes through the arc midpoint C .

Step 1. Since D is the midpoint of $\overline{AB}$, $AD = 3$. In right triangle ODA the hypotenuse is the radius $OA = 5$, so

$$OD = \sqrt{5^2 - 3^2} = \sqrt{16} = 4.$$

Step 2. The points O , D , C lie on one line, and $OC = 5$ is also a radius. So $DC = OC - OD = 5 - 4 = 1$.

Step 3. Triangle ADC has a right angle at D , with legs $AD = 3$ and $DC = 1$:

$$AC = \sqrt{3^2 + 1^2} = \sqrt{10}.$$

Watch out. 4 is the distance OD from the center to the chord, not the length AC . Stopping after Step 1 leads to choice (E).

Answer: (A) $\sqrt{10}$
2 Counting & Probability

Source: 2018 AMC 10A #11

When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n ?

- (A) 42 (B) 49 (C) 56 (D) 63 (E) 84

Key idea. All 6^7 outcomes are equally likely, so n is simply the number of ordered rolls with sum 10. Since every die shows at least 1, the seven dice already account for 7; only 3 “extra” pips are left to hand out — a tiny count.

Step 1. Write each roll as $1 + (\text{extra})$. The seven extras are whole numbers ≥ 0 that add up to $10 - 7 = 3$. No die can then show more than $1 + 3 = 4$, so the limit of 6 on a die never gets in the way.

Step 2. Split the 3 extras into cases by how they are grouped.

- All 3 on one die (one die shows 4): choose the die, 7 ways.
- 2 on one die and 1 on another (a 3 and a 2): choose the die showing 3 (7 ways), then a *different* die showing 2 (6 ways): $7 \cdot 6 = 42$ ways.
- 1 each on three dice (three 2's): choose the three dice, $\binom{7}{3} = 35$ ways.

Step 3. Add the cases: $n = 7 + 42 + 35 = 84$.

Watch out. In the middle case the two dice play different roles — one shows 3, the other 2. Counting it as $\binom{7}{2} = 21$ treats them as interchangeable, and the total becomes $7 + 21 + 35 = 63$, choice (D).

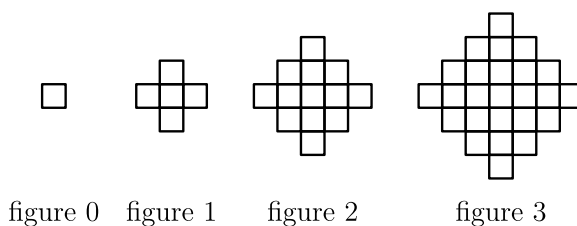
Another way. Stars and bars: place the 3 extras (stars) among 7 dice using 6 dividers (bars). That is 9 symbols, of which 3 are stars, so $n = \binom{9}{3} = 84$.

Answer: (E) 84

3 Algebra

Source: 2000 AMC 10 #12

Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 nonoverlapping unit squares, respectively. If the pattern were continued, how many nonoverlapping unit squares would there be in figure 100?



- (A) 10401 (B) 19801 (C) 20201 (D) 39801 (E) 40801

Key idea. Figure 100 is far too big to draw, so look for structure that grows in a predictable way. Read each figure column by column: the column heights are odd numbers rising to a peak and falling back.

Step 1. Look at figure 3 from left to right. Its columns have heights 1, 3, 5, 7, 5, 3, 1. Check: $1 + 3 + 5 + 7 + 5 + 3 + 1 = 25$. Figure 2 has columns 1, 3, 5, 3, 1 (total 13), and figure 1 has 1, 3, 1 (total 5).

Step 2. So figure n consists of the odd numbers $1, 3, \dots, 2n + 1$ going up, followed by $2n - 1, \dots, 3, 1$ coming down. The rising half is the first $n + 1$ odd numbers and the falling half is the first n odd numbers.

Step 3. The sum of the first m odd numbers is m^2 (for example $1 + 3 + 5 = 9$). Hence figure n has

$$(n + 1)^2 + n^2$$

squares. Check: $n = 0, 1, 2, 3$ give 1, 5, 13, 25. ✓

Step 4. For $n = 100$:

$$101^2 + 100^2 = 10201 + 10000 = 20201.$$

Watch out. The figures start at figure 0, so figure 100 is the 101st figure, not the 100th. Treating it as the 100th means using $n = 99$: $100^2 + 99^2 = 19801$, choice (B).

Another way. From one figure to the next the count grows by 4, 8, 12, $\dots$, that is, by $4k$ at step k . So figure 100 has $1 + 4(1 + 2 + \dots + 100) = 1 + 4 \cdot 5050 = 20201$ squares.

Answer: (C) 20201

4 Number Theory

Source: 2018 AMC 10B #13

How many of the first 2018 numbers in the sequence 101, 1001, 10001, 100001, $\dots$ are divisible by 101?

- (A) 253 (B) 504 (C) 505 (D) 506 (E) 1009

Key idea. Every term has the form $10^m + 1$, and $101 = 100 + 1$ is itself one of the terms. Since 100 is one less than 101, the remainders of powers of 10 on division by 101 must repeat in a short cycle — find the cycle.

Step 1. The k th term is $10^{k+1} + 1$, so as k runs from 1 to 2018, the exponent $m = k + 1$ runs from 2 to 2019.

Step 2. Work with remainders mod 101. Because $100 = 101 - 1$, we have $10^2 \equiv -1$. Then the powers of 10 cycle with period 4:

$$10^1 \equiv 10, \quad 10^2 \equiv -1, \quad 10^3 \equiv -10, \quad 10^4 \equiv 1,$$

and then the pattern repeats. (Indeed $10^4 - 1 = 9999 = 99 \cdot 101$.)

Step 3. Add 1: $10^m + 1$ leaves remainder 11, 0, -9 , 2 according as m leaves remainder 1, 2, 3, 0 when divided by 4. Only the case $m \equiv 2 \pmod{4}$ gives a multiple of 101. For example 101 ($m = 2$) and 100001 = $101 \cdot 9901$ ($m = 6$) work, while 1001, 10001, 100001 do not.

Step 4. Count m from 2 to 2019 with $m \equiv 2 \pmod{4}$: these are 2, 6, 10, $\dots$, 2018, which is

$$\frac{2018 - 2}{4} + 1 = 504 + 1 = 505$$

numbers.

Watch out. Computing $2018 \div 4 = 504.5$ and rounding down gives 504, choice (B). The list 2, 6, $\dots$, 2018 starts at the very first term, so the count is 505; always count an arithmetic list as $(\text{last} - \text{first}) \div \text{step} + 1$.

Answer: (C) 505

5 Algebra

Source: 2017 AMC 10A #14

Every week Roger pays for a movie ticket and a soda out of his allowance. Last week, Roger's allowance was A dollars. The cost of his movie ticket was 20% of the difference between A and the cost of his soda, while the cost of his soda was 5% of the difference between A and the cost of his movie ticket. To the nearest whole percent, what fraction of A did Roger pay for his movie ticket and soda?

- (A) 9% (B) 19% (C) 22% (D) 23% (E) 25%

Key idea. Two unknown prices are each described in terms of the other, so name them and write two equations. Only the *fraction* of A matters, so you may set $A = 1$ (or any convenient value).

Step 1. Let t be the ticket price and s the soda price, and take $A = 1$. (This is safe: doubling the allowance would double both prices, so the fraction of A they use does not depend on A .) The two sentences say

$$t = \frac{1}{5}(1 - s), \quad s = \frac{1}{20}(1 - t).$$

Step 2. Clear denominators: $5t = 1 - s$, so $s = 1 - 5t$. Substitute into $20s = 1 - t$:

$$20(1 - 5t) = 1 - t \implies 20 - 100t = 1 - t \implies 99t = 19 \implies t = \frac{19}{99}.$$

Step 3. Then

$$s = 1 - 5 \cdot \frac{19}{99} = \frac{99 - 95}{99} = \frac{4}{99}.$$

Check: $\frac{1}{5} \left(1 - \frac{4}{99} \right) = \frac{1}{5} \cdot \frac{95}{99} = \frac{19}{99}$. ✓

Step 4. The total is $t + s = \frac{23}{99} \approx 0.2323$, which is 23% to the nearest whole percent.

Watch out. The ticket is 20% of $A - s$, not 20% of A . Simply adding 20% + 5% gives 25%, choice (E). Since each price is a percent of something *smaller* than A , the true total must be below 25%.

Answer: (D) 23%

6 Number Theory

Source: 2015 AMC 10B #15

The town of Hamlet has 3 people for each horse, 4 sheep for each cow, and 3 ducks for each person. Which of the following could not possibly be the total number of people, horses, sheep, cows, and ducks in Hamlet?

- (A) 41 (B) 47 (C) 59 (D) 61 (E) 66

Key idea. Every count is tied by a ratio to either the horses or the cows, so write everything in terms of those two numbers. The total then has the form $13h + 5c$, and the question becomes: which choice cannot be written that way?

Step 1. Let h be the number of horses and c the number of cows (whole numbers). Then there are $3h$ people, $3 \cdot 3h = 9h$ ducks, and $4c$ sheep. The total is

$$h + 3h + 9h + c + 4c = 13h + 5c.$$

So the horses bring along a block of 13 animals-and-people, and each cow a block of 5.

Step 2. Four of the choices can be built from these blocks:

$$41 = 13 \cdot 2 + 5 \cdot 3, \quad 59 = 13 \cdot 3 + 5 \cdot 4, \quad 61 = 13 \cdot 2 + 5 \cdot 7, \quad 66 = 13 \cdot 2 + 5 \cdot 8.$$

Step 3. For 47, the horse blocks can only be $13h \leq 47$, i.e. $h = 0, 1, 2, 3$. What is left over for the cows would be

$$47, \quad 34, \quad 21, \quad 8,$$

and none of these is a multiple of 5. So no choice of h and c gives 47.

A quick way to scan: $5c$ always ends in 0 or 5, so $13h$ must end in the same digit as the total, or differ from it by 5. For 47 we need $13h$ to end in 7 or 2; the multiples 0, 13, 26, 39 end in 0, 3, 6, 9.

Answer: (B) 47

7 Algebra

Source: 2011 AMC 10A #16

Which of the following is equal to $\sqrt{9 - 6\sqrt{2}} + \sqrt{9 + 6\sqrt{2}}$?

- (A) $3\sqrt{2}$ (B) $2\sqrt{6}$ (C) $\frac{7\sqrt{2}}{2}$ (D) $3\sqrt{3}$ (E) 6

Key idea. The two radicands are *conjugates*, $9 \mp 6\sqrt{2}$. Squaring the whole sum makes the cross term a product of conjugates, which has no radical at all. (Alternatively, each radicand is a perfect square in disguise.)

Step 1. Let $S = \sqrt{9 - 6\sqrt{2}} + \sqrt{9 + 6\sqrt{2}}$. Note $S > 0$. Square it using $(x + y)^2 = x^2 + 2xy + y^2$:

$$S^2 = (9 - 6\sqrt{2}) + (9 + 6\sqrt{2}) + 2\sqrt{(9 - 6\sqrt{2})(9 + 6\sqrt{2})}.$$

Step 2. The first two terms add to 18. The product under the last root is a difference of squares:

$$(9 - 6\sqrt{2})(9 + 6\sqrt{2}) = 81 - 36 \cdot 2 = 81 - 72 = 9.$$

So $S^2 = 18 + 2\sqrt{9} = 18 + 6 = 24$.

Step 3. Since $S > 0$, $S = \sqrt{24} = 2\sqrt{6}$.

Check with decimals: $6\sqrt{2} \approx 8.485$, so the two roots are about $\sqrt{0.515} \approx 0.717$ and $\sqrt{17.485} \approx 4.182$. Their sum is about 4.899, and $2\sqrt{6} \approx 4.899$ as well.

Watch out. Square roots do not add: $\sqrt{x} + \sqrt{y} \neq \sqrt{x + y}$. “Combining” the two radicals into $\sqrt{18} = 3\sqrt{2}$ gives choice (A).

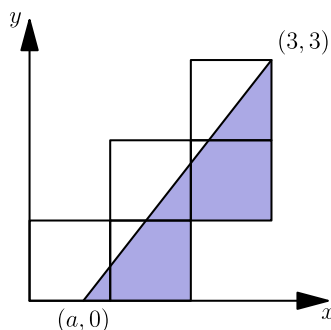
Another way. Look for $9 - 6\sqrt{2} = (\sqrt{p} - \sqrt{q})^2 = p + q - 2\sqrt{pq}$. We need $p + q = 9$ and $2\sqrt{pq} = 6\sqrt{2}$, i.e. $pq = 18$, so $p = 6$, $q = 3$. Thus $\sqrt{9 - 6\sqrt{2}} = \sqrt{6} - \sqrt{3}$ (positive, since $\sqrt{6} > \sqrt{3}$) and $\sqrt{9 + 6\sqrt{2}} = \sqrt{6} + \sqrt{3}$. Their sum is $2\sqrt{6}$.

Answer: (B) $2\sqrt{6}$

8 Geometry

Source: 2009 AMC 10B #17

Five unit squares are arranged in the coordinate plane as shown, with the lower left corner at the origin. The slanted line, extending from $(a, 0)$ to $(3, 3)$, divides the entire region into two regions of equal area. What is a ?

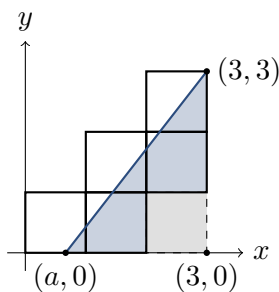


- (A) $\frac{1}{2}$ (B) $\frac{3}{5}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$ (E) $\frac{4}{5}$

Key idea. The shaded region is an awkward shape, but it is almost a right triangle: the triangle with vertices $(a, 0)$, $(3, 0)$, $(3, 3)$. The only difference is one missing unit square in the bottom-right corner. Compute the easy area and subtract.

Step 1. The five squares have total area 5, so the shaded region must have area $\frac{5}{2}$.

Step 2. Complete the shaded region to the right triangle with vertices $(a, 0)$, $(3, 0)$, $(3, 3)$. Its legs are $3 - a$ (along the x -axis) and 3 (vertical), so its area is $\frac{3(3-a)}{2}$.



Step 3. This triangle covers the shaded region plus the dashed square $[2, 3] \times [0, 1]$, which is *not* one of the five squares. So

$$\frac{3(3-a)}{2} - 1 = \frac{5}{2} \implies \frac{3(3-a)}{2} = \frac{7}{2} \implies 3-a = \frac{7}{3} \implies a = \frac{2}{3}.$$

Step 4. Check the picture: with $a = \frac{2}{3}$, at $x = 2$ the line is at height $3 \cdot \frac{2 - 2/3}{3 - 2/3} = \frac{12}{7}$, which is between 1 and 2. So the dashed square really does lie under the line, and the rest of the triangle lies inside the five squares.

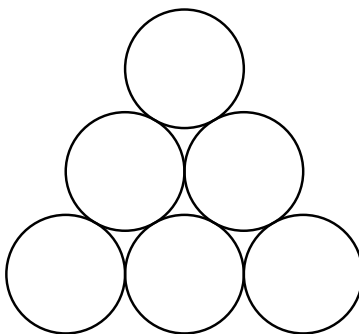
Watch out. Forgetting to remove the missing square gives $\frac{3(3-a)}{2} = \frac{5}{2}$, so $a = \frac{4}{3}$. That is not a choice, which is the hint that the triangle is bigger than the shaded region.

Answer: (C) $\frac{2}{3}$

9 Counting & Probability

Source: 2017 AMC 10B #18

In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible?



- (A) 6 (B) 8 (C) 9 (D) 12 (E) 15

Key idea. Symmetry problems get easier if you first place the color that appears only once. The single green disk can sit at only two essentially different places — a corner or the middle of a side — and once it is placed, very little symmetry remains.

Step 1. The figure is a triangle of disks: three *corner* disks and three *side* disks (the two in the middle row and the middle of the bottom row). Rotations and reflections move corners to corners and sides to sides, so a painting with green in a corner can never match one with green on a side. Rotating, we may assume green is at the top corner or at the bottom-middle disk.

Step 2. Put green at the top. There are $\binom{5}{2} = 10$ ways to choose the two red disks among the other five; the rest are blue. The only symmetry that keeps green at the top (besides doing nothing) is the left–right reflection. It fixes the bottom-middle disk, swaps the two middle-row disks, and swaps the two bottom corners.

Step 3. Of the 10 choices, exactly 2 are unchanged by the reflection: red on both middle-row disks, or red on both bottom corners. The other 8 come in mirror-image pairs, giving 4 more paintings. So green at the top gives $2 + 4 = 6$ paintings.

Step 4. With green at the bottom-middle, the same reasoning applies: the vertical reflection fixes the top disk and swaps two pairs, giving again $2 + 4 = 6$. In total, $6 + 6 = 12$.

Watch out. There are $\frac{6!}{3!2!1!} = 60$ colorings and 6 symmetries, but $60 \div 6 = 10$ is *not* the answer. Symmetric paintings have fewer than 6 distinct copies, so they are undercounted by that division.

Another way. Count copies. A painting fixed by a reflection appears only 3 times among the 60 colorings; there are 12 such colorings (green on a fixed disk: 2 ways; red on a swapped pair: 2 ways; for each of 3 reflections), giving $12 \div 3 = 4$ paintings. The remaining 48 colorings come 6 at a time: 8 paintings. Total $4 + 8 = 12$.

Answer: (D) 12

10 Geometry

Source: 2005 AMC 10A #20

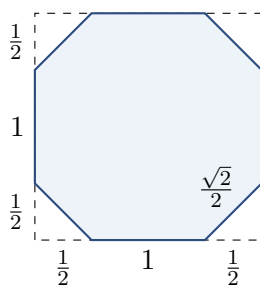
An equiangular octagon has four sides of length 1 and four sides of length $\frac{\sqrt{2}}{2}$, arranged so that no two consecutive sides have the same length. What is the area of the octagon?

- (A) $\frac{7}{2}$ (B) $\frac{7\sqrt{2}}{2}$ (C) $\frac{5+4\sqrt{2}}{2}$ (D) $\frac{4+5\sqrt{2}}{2}$ (E) 7

Key idea. Each interior angle of an equiangular octagon is 135° , so its sides point in only four directions: horizontal, vertical, and the two diagonals. A $\frac{\sqrt{2}}{2}$ length is exactly the diagonal of a $\frac{1}{2} \times \frac{1}{2}$ square — a strong hint to see the octagon as a square with its corners cut off.

Step 1. Each exterior angle is $360^\circ \div 8 = 45^\circ$, so consecutive sides turn by 45° . Since the lengths alternate, we can turn the octagon so that the four sides of length 1 are horizontal and vertical, and the four sides of length $\frac{\sqrt{2}}{2}$ run diagonally.

Step 2. Extend the four sides of length 1 until they meet. They form a square, and each diagonal side cuts off a corner triangle. That triangle has angles $45^\circ, 45^\circ, 90^\circ$ and hypotenuse $\frac{\sqrt{2}}{2}$, so each leg is $\frac{\sqrt{2}}{2} \div \sqrt{2} = \frac{1}{2}$.



Step 3. The side of the square is $\frac{1}{2} + 1 + \frac{1}{2} = 2$, so its area is 4. Each cut-off triangle has area $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$, and there are four of them:

$$\text{Area} = 4 - 4 \cdot \frac{1}{8} = 4 - \frac{1}{2} = \frac{7}{2}.$$

Another way. Cut the octagon into pieces instead: a central 1×1 square (area 1), four $1 \times \frac{1}{2}$ rectangles on its sides (area $4 \cdot \frac{1}{2} = 2$), and four corner triangles with legs $\frac{1}{2}$ (area $4 \cdot \frac{1}{8} = \frac{1}{2}$). Total $1 + 2 + \frac{1}{2} = \frac{7}{2}$.

Answer: (A) $\frac{7}{2}$
