
AMC 10

SET 6

HALF-LENGTH PRACTICE

SOLUTIONS

[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	C	2012 AMC 10A #10	Algebra
2	C	2019 AMC 10A #11	Number Theory
3	B	2013 AMC 10A #13	Counting & Probability
4	A	2009 AMC 10A #14	Geometry
5	D	2004 AMC 10A #15	Algebra
6	E	2007 AMC 10A #16	Counting & Probability
7	B	2004 AMC 10B #17	Algebra
8	C	2016 AMC 10A #18	Counting & Probability
9	E	2002 AMC 10A #19	Geometry
10	B	2017 AMC 10B #20	Number Theory

1 Algebra

Source: 2012 AMC 10A #10

Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

- (A) 5 (B) 6 (C) 8 (D) 10 (E) 12

Key idea. Write the twelve angles as $a, a + d, \dots, a + 11d$ and use the fact that they fill the whole circle: their sum is 360.

The sum of an arithmetic sequence is (number of terms) $\times$ (average of first and last), so

$$\frac{12}{2}(a + (a + 11d)) = 12a + 66d = 360.$$

Dividing by 6 gives $2a + 11d = 60$.

To make a small we want d as large as possible. Since $2a$ and 60 are even, $11d$ must be even, so d is even. Also $a > 0$ forces $11d < 60$, so $d \leq 5$. The largest even choice is $d = 4$, and then $2a = 60 - 44 = 16$, so $a = 8$.

Check: the angles 8, 12, 16, $\dots$, 52 are integers and $12a + 66d = 96 + 264 = 360$.

Another way. Test the choices from the smallest. With $a = 5$: $11d = 50$; with $a = 6$: $11d = 48$ — neither is a multiple of 11. With $a = 8$: $11d = 44$, $d = 4$. It works.

Answer: (C) 8

2 Number Theory

Source: 2019 AMC 10A #11

How many positive integer divisors of 201^9 are perfect squares or perfect cubes (or both)?

- (A) 32 (B) 36 (C) 37 (D) 39 (E) 41

Key idea. Counting divisors with a property is really counting *exponent choices*, so factor 201^9 first. The words “or both” are a signal for inclusion–exclusion: count squares, count cubes, then subtract what was counted twice.

Step 1. $201 = 3 \cdot 67$, so $201^9 = 3^9 \cdot 67^9$. Every divisor has the form $3^i \cdot 67^j$ with $0 \leq i, j \leq 9$.

Step 2. A number is a perfect square exactly when every exponent in its prime factorization is even. The even values in $0, \dots, 9$ are 0, 2, 4, 6, 8 — five of them — so there are $5 \cdot 5 = 25$ square divisors.

Step 3. A perfect cube needs every exponent divisible by 3: the options are 0, 3, 6, 9, four of them, giving $4 \cdot 4 = 16$ cube divisors.

Step 4. A divisor that is both a square and a cube is a sixth power: each exponent is a multiple of 6, so 0 or 6. That gives $2 \cdot 2 = 4$ divisors, and each was counted in both Step 2 and Step 3.

By inclusion–exclusion, the count is $25 + 16 - 4 = 37$.

Watch out. Adding $25 + 16 = 41$ counts the four sixth powers (such as 1 and 3^6) twice and leads to (E).

Answer: (C) 37

3 Counting & Probability

Source: 2013 AMC 10A #13

How many three-digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

- (A) 52 (B) 60 (C) 66 (D) 68 (E) 70

Key idea. The number has the shape $\overline{xyx}$, so there are only two digits to choose. Three conditions on two digits suggests organizing a short case count by the outer digit x .

Step 1. Write the number as $\overline{xyx}$, with hundreds digit and units digit both equal to x , and tens digit y . Being a three-digit number means $x \neq 0$.

Step 2. A number is divisible by 5 exactly when its units digit is 0 or 5. The units digit is x , so $x \neq 5$ (and $x \neq 0$ already). That leaves $x \in \{1, 2, 3, 4, 6, 7, 8, 9\}$.

Step 3. The digit sum is $x + y + x = 2x + y$, and we need $2x + y < 20$, i.e. $y < 20 - 2x$, with $0 \leq y \leq 9$.

- $x = 1, 2, 3, 4$: $20 - 2x \geq 12$, so every y from 0 to 9 works — 10 each, 40 total.
- $x = 6$: $y < 8$, so $y = 0, \dots, 7$ — 8 choices.
- $x = 7$: $y < 6$ — 6 choices.
- $x = 8$: $y < 4$ — 4 choices.
- $x = 9$: $y < 2$ — 2 choices.

Step 4. Total: $40 + 8 + 6 + 4 + 2 = 60$.

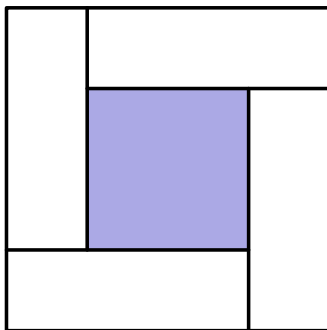
Watch out. Forgetting the “not divisible by 5” condition lets $x = 5$ back in. Then $2x + y = 10 + y < 20$ for all 10 values of y , and the count becomes 70, choice (E).

Answer: (B) 60

4 Geometry

Source: 2009 AMC 10A #14

Four congruent rectangles are placed as shown. The area of the outer square is 4 times that of the inner square. What is the ratio of the length of the longer side of each rectangle to the length of its shorter side?



- (A) 3 (B) $\sqrt{10}$ (C) $2 + \sqrt{2}$ (D) $2\sqrt{3}$ (E) 4

Key idea. Name the two sides of one rectangle and read both square sides off the picture in terms of them. An *area* ratio of 4 is a *side* ratio of 2, which turns the condition into one linear equation.

Step 1. Let each rectangle have shorter side x and longer side y . Along the top edge of the outer square you see the short side of the left rectangle followed by the long side of the top rectangle, so the outer square has side $y + x$.

Step 2. Now cut straight across the middle of the figure. You pass through the left rectangle (width x), the inner square, and the right rectangle (width x). So the inner square has side $(y + x) - 2x = y - x$.

Step 3. Areas scale as the square of the side, so area ratio 4 means side ratio $\sqrt{4} = 2$:

$$y + x = 2(y - x) \implies y + x = 2y - 2x \implies y = 3x.$$

So $\frac{y}{x} = 3$.

Check with $x = 1$, $y = 3$: the outer side is 4 (area 16), the inner side is 2 (area 4), and $16 = 4 \cdot 4$.

Watch out. Using the area ratio as a side ratio, $y + x = 4(y - x)$, gives $\frac{y}{x} = \frac{5}{3}$ — not among the choices, a sign that the square root was skipped.

Answer: (A) 3

5 Algebra

Source: 2004 AMC 10A #15

Given that $-4 \leq x \leq -2$ and $2 \leq y \leq 4$, what is the largest possible value of

$$\frac{x+y}{x}?$$

- (A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

Key idea. The variable x appears both on top and on the bottom, which makes it hard to see which way to push it. Splitting the fraction as $1 + \frac{y}{x}$ leaves a single simple ratio to optimize.

Step 1. Divide each term of the numerator by x :

$$\frac{x+y}{x} = \frac{x}{x} + \frac{y}{x} = 1 + \frac{y}{x}.$$

So we need the largest possible value of $\frac{y}{x}$.

Step 2. Here y is positive and x is negative, so $\frac{y}{x}$ is always negative. A negative number is largest when it is closest to 0, that is, when its size $\frac{|y|}{|x|}$ is as small as possible.

Step 3. To make $\frac{|y|}{|x|}$ small, take the numerator as small as possible, $y = 2$, and the denominator as large as possible, $|x| = 4$, i.e. $x = -4$. Then $\frac{y}{x} = \frac{2}{-4} = -\frac{1}{2}$.

Step 4. The largest value is $1 - \frac{1}{2} = \frac{1}{2}$. Check directly: $\frac{-4+2}{-4} = \frac{-2}{-4} = \frac{1}{2}$.

Watch out. Making the numerator $x + y$ as big as possible — $x = -2$ with $y = 4$ — ignores that the denominator is negative. It makes $\frac{y}{x} = -2$ and the expression $1 - 2 = -1$. That is the *smallest* possible value, and it is choice (A).

Answer: (D) $\frac{1}{2}$

6 Counting & Probability

Source: 2007 AMC 10A #16

Integers a , b , c , and d , not necessarily distinct, are chosen independently and at random from 0 to 2007, inclusive. What is the probability that $ad - bc$ is even?

- (A) $\frac{3}{8}$ (B) $\frac{7}{16}$ (C) $\frac{1}{2}$ (D) $\frac{9}{16}$ (E) $\frac{5}{8}$

Key idea. Whether $ad - bc$ is even depends only on whether each number is odd or even. That reduces the problem to coin flips — and a difference is even exactly when both parts have the same parity.

Step 1. From 0 to 2007 there are 2008 integers: 1004 even and 1004 odd. So each of a, b, c, d is odd with probability exactly $\frac{1}{2}$, independently.

Step 2. A product is odd only when both factors are odd. So

$$P(ad \text{ odd}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}, \quad P(ad \text{ even}) = 1 - \frac{1}{4} = \frac{3}{4},$$

and the same holds for bc . Since ad uses a, d and bc uses b, c , the two products are independent.

Step 3. $ad - bc$ is even when ad and bc are both even or both odd:

$$\frac{3}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{9}{16} + \frac{1}{16} = \frac{10}{16} = \frac{5}{8}.$$

Watch out. Counting only the case where both products are even gives $\frac{9}{16}$, choice (D). Odd minus odd is also even — don't forget that case.

Answer: (E) $\frac{5}{8}$

7 Algebra

Source: 2004 AMC 10B #17

The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years Jack will be twice as old as Bill will be then. What is the difference in their current ages?

- (A) 9 (B) 18 (C) 27 (D) 36 (E) 45

Key idea. "Digits in reverse order" means writing each age in place-value form, $10x + y$ and $10y + x$. The age condition then becomes one equation in two *digits*, and the tight range 0–9 lets you find them by checking a handful of cases.

Step 1. Let Jack's age be $10x + y$ and Bill's age $10y + x$, where x and y are digits. In five years Jack is twice as old as Bill:

$$10x + y + 5 = 2(10y + x + 5) = 20y + 2x + 10.$$

Collecting terms gives $8x = 19y + 5$.

Step 2. Since $x \leq 9$, the left side is at most 72, so $19y + 5 \leq 72$ and $y \leq 3$. Try each value:

- $y = 0$: $19y + 5 = 5$, not a multiple of 8.
- $y = 1$: $19y + 5 = 24 = 8 \cdot 3$, so $x = 3$.
- $y = 2$: $19y + 5 = 43$, not a multiple of 8.
- $y = 3$: $19y + 5 = 62$, not a multiple of 8.

So the only solution is $x = 3$, $y = 1$: Jack is 31 and Bill is 13.

Step 3. Check: in five years they are 36 and 18, and $36 = 2 \cdot 18$. The difference in their ages is $31 - 13 = 18$.

Notice that the difference is always $(10x + y) - (10y + x) = 9(x - y)$, a multiple of 9. Every choice is a multiple of 9, so this alone cannot pick the answer — you really must find $x - y = 2$.

Answer: (B) 18

8 Counting & Probability

Source: 2016 AMC 10A #18

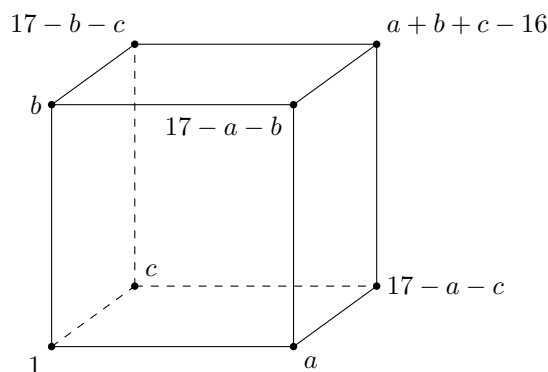
Each vertex of a cube is to be labeled with an integer 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?

- (A) 1 (B) 3 (C) 6 (D) 12 (E) 24

Key idea. First find the common face sum: two opposite faces together use every vertex once. Then notice that a face sum lets you fill in a face's fourth corner from the other three — so once 1 and its three neighbors are placed, the whole cube is forced. Count those choices, then deal with rotations.

Step 1. Two opposite faces share no vertex and together contain all eight, so their sums add to $1 + 2 + \cdots + 8 = 36$. Each face therefore sums to 18.

Step 2. Rotate the cube so that 1 is at the front-bottom-left corner, and call its three neighbors a , b , c . Each face through 1 has one unknown corner, which is forced:



For example, the front face gives $18 - 1 - a - b = 17 - a - b$. The right face then gives the far corner: $18 - a - (17 - a - b) - (17 - a - c) = a + b + c - 16$. The last two faces (back and top) then sum to 18 automatically.

Step 3. The far corner is at least 2, so $a + b + c \geq 18$, with a, b, c from $2, \dots, 8$. Only seven sets qualify; we need the four forced labels to be exactly the four unused numbers:

$\{a, b, c\}$	forced labels	valid?
$\{3, 7, 8\}$	7, 6, 2, 2	no
$\{5, 6, 7\}$	6, 5, 4, 2	no
$\{4, 6, 8\}$	7, 5, 3, 2	yes
$\{4, 7, 8\}$	6, 5, 2, 3	yes
$\{5, 6, 8\}$	6, 4, 3, 3	no
$\{5, 7, 8\}$	5, 4, 2, 4	no
$\{6, 7, 8\}$	4, 3, 2, 5	yes

Step 4. Each valid set can be placed on the three neighbors in $3! = 6$ orders. The only rotations that keep 1 in its corner are turns by 120° and 240° about the long diagonal through it, which cycle the three neighbors. So the 6 orders fall into 2 genuinely different groups (the two mirror-image cyclic orders), giving $3 \cdot 2 = 6$ arrangements.

Watch out. Counting each valid neighbor set once gives 3, choice (B). A mirror image is *not* a rotation, so each set yields two arrangements.

Answer: (C) 6

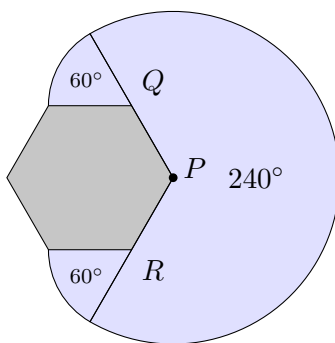
9 Geometry

Source: 2002 AMC 10A #19

Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the area, in square yards, of the region outside the doghouse that Spot can reach?

- (A) $\frac{2}{3}\pi$ (B) 2π (C) $\frac{5}{2}\pi$ (D) $\frac{8}{3}\pi$ (E) 3π

Key idea. A tethered animal sweeps out circular sectors. The question is what happens when the rope hits the house: it bends around a corner and Spot keeps going on a shorter rope. Since the rope (2) is longer than a side (1), expect extra small sectors past the neighboring corners.



Step 1. Each interior angle of a regular hexagon is 120° . At the tether point P the house blocks that 120° , so Spot can swing his full 2-yard rope through $360^\circ - 120^\circ = 240^\circ$. That sector has area

$$\frac{240}{360} \cdot \pi \cdot 2^2 = \frac{2}{3} \cdot 4\pi = \frac{8\pi}{3}.$$

Step 2. When the rope lies along side PQ , 1 yard of it is used up and 1 yard remains beyond Q . Pivoting at Q , Spot can swing until the rope lies along the next side. The exterior angle of the hexagon is $180^\circ - 120^\circ = 60^\circ$, so this is a 60° sector of radius 1:

$$\frac{60}{360} \cdot \pi \cdot 1^2 = \frac{\pi}{6}.$$

The same happens around R . These pieces do not overlap the big sector, and the rope is then exactly used up at the next corners.

Step 3. Total area: $\frac{8\pi}{3} + 2 \cdot \frac{\pi}{6} = \frac{8\pi}{3} + \frac{\pi}{3} = 3\pi$.

Watch out. Stopping after the big sector gives $\frac{8}{3}\pi$, choice (D). The rope is longer than a side, so it wraps around the neighboring corners.

Answer: (E) 3π

10 Number Theory

Source: 2017 AMC 10B #20

The number $21! = 51,090,942,171,709,440,000$ has over 60,000 positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

- (A) $\frac{1}{21}$ (B) $\frac{1}{19}$ (C) $\frac{1}{18}$ (D) $\frac{1}{2}$ (E) $\frac{11}{21}$

Key idea. A divisor is odd exactly when it contains no factor of 2. So the only prime that matters is 2: find its exponent in $21!$, and the big number printed in the problem is a distraction.

Step 1. Count the factors of 2 in $21! = 1 \cdot 2 \cdot 3 \cdots 21$. There are $\lfloor 21/2 \rfloor = 10$ multiples of 2, each contributing one factor; $\lfloor 21/4 \rfloor = 5$ multiples of 4 each contribute a second; $\lfloor 21/8 \rfloor = 2$ multiples of 8 a third; and $\lfloor 21/16 \rfloor = 1$ multiple of 16 a fourth. In total

$$10 + 5 + 2 + 1 = 18,$$

so $21! = 2^{18} \cdot m$ with m odd.

Step 2. Every divisor of $21!$ can be written uniquely as $2^k \cdot t$, where t is a divisor of m and $0 \leq k \leq 18$. So the divisors split into groups, one for each t :

$$t, 2t, 4t, \dots, 2^{18}t \quad (19 \text{ divisors}).$$

Step 3. In each group exactly one divisor — the one with $k = 0$ — is odd. All groups have the same size 19, so exactly $\frac{1}{19}$ of all divisors are odd.

(Check: $21!$ has 60,800 divisors, of which 3,200 are odd, and $3200/60800 = 1/19$.)

Watch out. There are 19 possible exponents $k = 0, 1, \dots, 18$, not 18. Using 18 gives $\frac{1}{18}$, choice (C). And “half of all numbers are odd” does not apply to divisors: guessing $\frac{1}{2}$ is choice (D).

Answer: (B) $\frac{1}{19}$