# AMC 10

SET 7

HALF-LENGTH PRACTICE

# SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

## Answer Key and Sources

| No. | Ans. | Source | Topic |
|---|---|---|---|
| 1 | **C** | 2015 AMC 10B #10 | Number Theory |
| 2 | **B** | 2008 AMC 10B #11 | Algebra |
| 3 | **C** | 2004 AMC 10A #12 | Counting & Probability |
| 4 | **C** | 2009 AMC 10B #13 | Geometry |
| 5 | **C** | 2014 AMC 10A #15 | Algebra |
| 6 | **B** | 2017 AMC 10A #16 | Number Theory |
| 7 | **D** | 2016 AMC 10B #17 | Geometry |
| 8 | **D** | 2009 AMC 10A #18 | Algebra |
| 9 | **E** | 2016 AMC 10A #19 | Geometry |
| 10 | **D** | 2011 AMC 10A #20 | Counting & Probability |

## 1 Number Theory

Source: 2015 AMC 10B #10

What are the sign and units digit of the product of all the odd negative integers strictly greater than $-2015$?

**(A)** It is a negative number ending with a 1.
**(B)** It is a positive number ending with a 1.
**(C)** It is a negative number ending with a 5.
**(D)** It is a positive number ending with a 5.
**(E)** It is a negative number ending with a 0.

**Key idea.** Settle the sign and the last digit separately: the sign depends only on *how many* negative factors there are, and the last digit is settled by asking whether a factor of 5 and a factor of 2 appear.

**Sign.** The numbers are $-1, -3, -5, \ldots, -2013$ ($-2015$ itself is excluded by "strictly"). Their absolute values $1, 3, \ldots, 2013$ are the odd numbers up to 2013, and there are $\frac{2013+1}{2} = 1007$ of them. An odd number of negative factors gives a negative product.

**Units digit.** The factor $-5$ is in the list, so the product is a multiple of 5 and ends in 0 or 5. Every factor is odd, so the product is odd and cannot end in 0. It ends in 5.

**Watch out.** Dropping the word "strictly" adds $-2015$ and makes 1008 factors, an even count — that gives a positive number ending in 5, choice (D). And "multiple of 5" alone does not mean "ends in 0": with no even factor it must end in 5, so (E) is out.

**Answer: (C) negative, ending in 5**

## 2 Algebra

Source: 2008 AMC 10B #11

Suppose that $(u_n)$ is a sequence of real numbers satisfying $u_{n+2} = 2u_{n+1} + u_n$, and that $u_3 = 9$ and $u_6 = 128$. What is $u_5$?

**(A)** 40 **(B)** 53 **(C)** 68 **(D)** 88 **(E)** 104

**Key idea.** The two known terms, $u_3$ and $u_6$, are three steps apart, with $u_4$ and $u_5$ missing in between. Name the first missing term $u_4$, write the next terms in terms of it, and the value $u_6 = 128$ gives one equation for it.

**Step 1.** Let $u_4 = x$. Use the rule with $n = 3$:

$$u_5 = 2u_4 + u_3 = 2x + 9.$$

**Step 2.** Use the rule again with $n = 4$:

$$u_6 = 2u_5 + u_4 = 2(2x + 9) + x = 5x + 18.$$

**Step 3.** Since $u_6 = 128$, we get $5x + 18 = 128$, so $5x = 110$ and $x = 22$. That is, $u_4 = 22$.

**Step 4.** Then $u_5 = 2 \cdot 22 + 9 = 44 + 9 = 53$.

*Check:* $u_6 = 2 \cdot 53 + 22 = 106 + 22 = 128$. ✓

Notice that we never needed $u_1$ or $u_2$: the recurrence only links terms that are next to each other, so the gap between $u_3$ and $u_6$ is all that matters.

**Watch out.** Solving for $x$ answers the question about $u_4$, not $u_5$. Stopping at $x = 22$ (not a choice) is a signal to take one more step.

**Answer: (B)** 53

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## 3 Counting & Probability

Source: 2004 AMC 10A #12

> Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?
>
> **(A)** 24 **(B)** 256 **(C)** 768 **(D)** 40,320 **(E)** 120,960

**Key idea.** A hamburger is fixed by a string of independent yes/no decisions — one for each condiment — plus a choice of patty count. Independent choices multiply.

**Step 1: condiments.** Go down the list of 8 condiments and decide for each one: on the burger or not. That is 2 options per condiment, and the decisions do not affect one another, so the number of condiment collections is

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256.$$

This count includes the empty collection (a plain burger), which is allowed: "any collection" means any, including none.

**Step 2: patties.** For each of those 256 collections the customer chooses 1, 2, or 3 patties — 3 options.

**Step 3.** The total number of different hamburgers is

$$3 \times 256 = 768.$$

Order does not matter here: a burger with ketchup then mustard is the same burger as one with mustard then ketchup. That is why we count *subsets* ($2^8$), not arrangements.

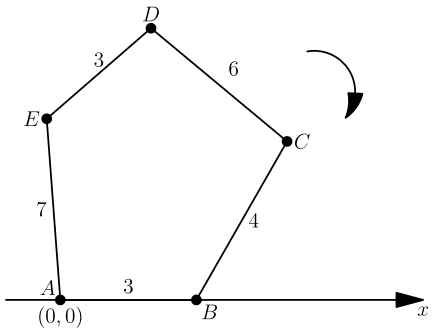
**Watch out.** Counting orderings of the condiments gives $8! = 40{,}320$, choice (D), and multiplying that by 3 gives 120,960, choice (E) — but a burger is a set of toppings, not a sequence. Forgetting the patties leaves 256, choice (B).

**Answer: (C)** 768

## 4 Geometry

Source: 2009 AMC 10B #13

As shown below, convex pentagon $ABCDE$ has sides $AB = 3$, $BC = 4$, $CD = 6$, $DE = 3$, and $EA = 7$. The pentagon is originally positioned in the plane with vertex $A$ at the origin and vertex $B$ on the positive $x$-axis. The pentagon is then rolled clockwise to the right along the $x$-axis. Which side will touch the point $x = 2009$ on the $x$-axis?




**(A)** $\overline{AB}$ **(B)** $\overline{BC}$ **(C)** $\overline{CD}$ **(D)** $\overline{DE}$ **(E)** $\overline{EA}$

**Key idea.** When a polygon rolls without slipping, each side lays itself down on the axis in turn, so the contact points just add up the side lengths. The number 2009 is far too big to roll by hand, which says: find the period (the perimeter) and work with the remainder.

**Step 1: the order of the sides.** At the start $\overline{AB}$ covers $[0, 3]$. Rolling clockwise pivots the pentagon about $B$, so $C$ comes down next and $\overline{BC}$ is the next side on the axis; after that come $\overline{CD}$, $\overline{DE}$, $\overline{EA}$, and then $\overline{AB}$ again.

**Step 2: one full turn.** The perimeter is $3 + 4 + 6 + 3 + 7 = 23$. After every complete turn, vertex $A$ is back on the axis 23 units further right, at $0, 23, 46, \ldots$

**Step 3: reduce 2009.** Since $23 \times 87 = 2001$, we have $2009 = 2001 + 8$. So after 87 full turns $A$ is at $x = 2001$, and we need to know which side covers the point 8 units beyond it.

**Step 4: walk the remainder.** Starting from $A$:

$$\overline{AB} : [0, 3], \quad \overline{BC} : [3, 7], \quad \overline{CD} : [7, 13].$$

The offset 8 lies strictly inside $[7, 13]$, so side $\overline{CD}$ covers it. In actual coordinates, $C$ lands at 2008 and $D$ at 2014, and $2008 < 2009 < 2014$.

**Watch out.** The sides have different lengths, so you cannot just count sides. Reducing 2009 modulo 5 (remainder 4) and taking the fourth side gives $\overline{DE}$, choice (D) — wrong, because the step size is length, not one side at a time.

**Answer: (C)** $\overline{CD}$

## 5 Algebra

Source: 2014 AMC 10A #15

> David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?
>
> **(A)** 140 **(B)** 175 **(C)** 210 **(D)** 245 **(E)** 280

**Key idea.** Both scenarios describe the same trip and the same deadline. Write each trip time in terms of the distance $d$; "1 hour late" versus "30 minutes early" tells you exactly how far apart those two times are.

**Step 1: the slow plan.** At 35 miles per hour the whole trip would take $\frac{d}{35}$ hours, and that is 1 hour *more* than the time he has.

**Step 2: the actual trip.** The first hour covers 35 miles. The remaining $d-35$ miles go at $35+15=50$ miles per hour, taking $\frac{d-35}{50}$ hours. So the actual time is $1+\frac{d-35}{50}$, and that is $\frac{1}{2}$ hour *less* than the time he has.

**Step 3: compare.** The slow plan is $1+\frac{1}{2}=\frac{3}{2}$ hours longer than the actual trip:

$$\frac{d}{35}-\left(1+\frac{d-35}{50}\right)=\frac{3}{2}.$$

Multiply by 350 (the least common multiple of 35, 50, 2):

$$10d-350-7(d-35)=525,$$

so $10d-350-7d+245=525$, giving $3d-105=525$, $3d=630$, and $d=210$.

*Check:* at 35 mph the trip takes 6 hours; the actual trip takes $1+\frac{175}{50}=4.5$ hours. The difference is 1.5 hours: 1 hour late versus 30 minutes early. ✓

**Watch out.** The first hour is driven at 35 mph in *both* scenarios. Pretending the whole trip is at 50 mph, $\frac{d}{35}-\frac{d}{50}=\frac{3}{2}$, gives $d=175$, choice (B).

**Answer: (C)** 210

## 6 Number Theory

Source: 2017 AMC 10A #16

> There are 10 horses, named Horse 1, Horse 2, ..., Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse $k$ runs one lap in exactly $k$ minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds.
>
> The least time $S > 0$, in minutes, at which all 10 horses will again simultaneously be at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that at least 5 of the horses are again at the starting point. What is the sum of the digits of $T$?
>
> **(A)** 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

**Key idea.** The given value $S = 2520 = \text{lcm}(1, \ldots, 10)$ shows the translation: Horse $k$ is at the start exactly at multiples of $k$. So "at least 5 horses at the start" means "divisible by at least 5 of the numbers 1 through 10", and we want the smallest such time.

**Step 1: translate.** Horse $k$ finishes laps at times $k, 2k, 3k, \ldots$, so it is at the starting point at time $t$ exactly when $k$ divides $t$. We need the least positive integer $t$ with at least five divisors among $1, 2, \ldots, 10$. (Between whole minutes no horse is back at the start, so only whole numbers matter.)

**Step 2: check small $t$.** List the divisors from $\{1, \ldots, 10\}$:

- $t = 6$: $1, 2, 3, 6$ — four horses.
- $t = 8$: $1, 2, 4, 8$ — four.
- $t = 10$: $1, 2, 5, 10$ — four.
- $t = 12$: $1, 2, 3, 4, 6$ — five.

Every other $t$ below 12 does no better: $1, 2, 3, 4, 5, 7, 9, 11$ have at most three such divisors. So $T = 12$.

**Step 3.** The sum of the digits of $T$ is $1 + 2 = 3$.

**Watch out.** Horse 1 is at the start every minute, and it counts as one of the five. Leaving it out forces you to look for five divisors among $2, \ldots, 10$, which first happens at 24 (divisors $2, 3, 4, 6, 8$), giving digit sum 6, choice (E).

**Answer: (B)** 3

## 7 Geometry

Source: 2016 AMC 10B #17

> All the numbers $2, 3, 4, 5, 6, 7$ are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, a product of three numbers is computed, where the three numbers are the numbers assigned to the three faces that include that vertex. What is the greatest possible value of the sum of these eight products?
>
> **(A)** 312 **(B)** 343 **(C)** 625 **(D)** 729 **(E)** 1680

**Key idea.** A sum of eight products with one factor from each of three "pairs" is exactly what you get when you expand a product of three sums. Spotting that factorization turns the problem into maximizing a product with a fixed sum.

**Step 1: factor the sum.** Split the six faces into three pairs of opposite faces, labelled $a_1, a_2$; $b_1, b_2$; $c_1, c_2$. A vertex touches exactly one face from each opposite pair (it cannot touch two opposite faces), and every choice of one face from each pair meets at a vertex — $2 \times 2 \times 2 = 8$ choices for 8 vertices. So the eight vertex products are exactly the eight terms of

$$(a_1 + a_2)(b_1 + b_2)(c_1 + c_2).$$

**Step 2: the constraint.** The three pair-sums add up to $2 + 3 + 4 + 5 + 6 + 7 = 27$.

**Step 3: the bound.** For positive numbers with a fixed total, the product is largest when they are equal (AM–GM): $xyz \le \left(\frac{x+y+z}{3}\right)^3 = 9^3 = 729$.

**Step 4: it is reachable.** Put opposite each other 2 and 7, 3 and 6, 4 and 5. Each pair sums to 9, and the total is $9 \cdot 9 \cdot 9 = 729$.

So 729 is both an upper bound and achieved.

Notice that only the three opposite-pair sums matter, not which pairs end up next to each other. An unbalanced pairing such as 7 with 6, 5 with 4, 3 with 2 gives just $13 \cdot 9 \cdot 5 = 585$.

**Answer: (D)** 729

## 8 Algebra

Source: 2009 AMC 10A #18

> At Jefferson Summer Camp, 60% of the children play soccer, 30% of the children swim, and 40% of the soccer players swim. To the nearest whole percent, what percent of the non-swimmers play soccer?
>
> **(A)** 30% **(B)** 40% **(C)** 49% **(D)** 51% **(E)** 70%

**Key idea.** Percents of different groups are hard to juggle, so pick a concrete camp of 100 children and turn every percent into a head count. The question's phrase "percent *of the non-swimmers*" tells you the denominator is the number of non-swimmers, not 100.

**Step 1: soccer players.** Of 100 children, 60% play soccer: 60 children.

**Step 2: soccer players who swim.** 40% of those 60 swim: $0.4 \times 60 = 24$ children. So $60 - 24 = 36$ soccer players do *not* swim.

**Step 3: non-swimmers.** 30% of the children swim, so 30 swim and $100 - 30 = 70$ do not.

**Step 4: the fraction.** Among the 70 non-swimmers, 36 play soccer:

$$\frac{36}{70} = \frac{18}{35} \approx 0.514,$$

which is 51% to the nearest whole percent.

Using 100 children loses nothing: every count scales with the size of the camp, so the ratio $\frac{36}{70}$ is the same for any camp.

**Watch out.** The denominator is where this goes wrong. Dividing the 36 non-swimming soccer players by all 100 children gives 36%, and dividing by the 60 soccer players gives 60% — neither answers "percent of the non-swimmers".

**Answer: (D)** 51%

## 9 Geometry

Source: 2016 AMC 10A #19

> In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point $E$ between $B$ and $C$, and point $F$ between $E$ and $C$ are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at $P$ and $Q$, respectively.
>
> The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of $r$, $s$, and $t$ is 1. What is $r + s + t$?
>
> **(A)** 7 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 20

**Key idea.** Every point in question sits where a line from $A$ crosses the diagonal $\overline{BD}$, and $\overline{AD}$ is parallel to $\overline{BC}$. Two parallel segments joined by crossing lines form a "bow-tie" of similar triangles, which hands you the ratios along $\overline{BD}$ directly.

**Step 1: lengths.** $BC = 3$ and $BE = EF = FC$, so $BE = 1$, $BF = 2$. Also $AD = BC = 3$.

**Step 2: locate $P$.** Since $\overline{BE} \parallel \overline{AD}$, triangles $PBE$ and $PDA$ are similar (vertical angles at $P$, alternate angles at $B$ and $D$). Hence

$$\frac{BP}{PD} = \frac{BE}{DA} = \frac{1}{3}, \qquad \text{so} \qquad BP = \frac{1}{4}\,BD.$$

**Step 3: locate $Q$.** In the same way, triangles $QBF$ and $QDA$ are similar, so

$$\frac{BQ}{QD} = \frac{BF}{DA} = \frac{2}{3}, \qquad \text{so} \qquad BQ = \frac{2}{5}\,BD, \quad QD = \frac{3}{5}\,BD.$$

**Step 4: the three pieces.** $PQ = BQ - BP = \left(\frac{2}{5} - \frac{1}{4}\right) BD = \frac{3}{20}\,BD$. Therefore

$$BP : PQ : QD = \frac{1}{4} : \frac{3}{20} : \frac{3}{5} = \frac{5}{20} : \frac{3}{20} : \frac{12}{20} = 5 : 3 : 12.$$

The numbers 5, 3, 12 have greatest common factor 1, so $r + s + t = 5 + 3 + 12 = 20$.

**Watch out.** $\frac{BP}{PD} = \frac{1}{3}$ compares $BP$ with $PD$, not with the whole diagonal: $BP$ is $\frac{1}{1+3} = \frac{1}{4}$ of $BD$. Also, $\frac{2}{5}BD$ is the distance from $B$ to $Q$; you must subtract $BP$ to get $PQ$.

**Answer: (E)** 20

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## 10 Counting & Probability

Source: 2011 AMC 10A #20

Two points on the circumference of a circle of radius $r$ are selected independently and at random. From each point a chord of length $r$ is drawn in a clockwise direction. What is the probability that the two chords intersect?

**(A)** $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

**Key idea.** A chord of length $r$ in a circle of radius $r$ forms an equilateral triangle with the center, so it always cuts off a $60°$ arc. Because of rotational symmetry, you can fix the first chord and ask only where the second chord may start.

**Step 1: the arc.** The chord, together with the two radii to its ends, makes a triangle with all sides $r$ — equilateral — so the central angle is $60°$. Each chord therefore spans a $60°$ arc, running clockwise from its starting point.

**Step 2: fix the first chord.** Rotate so that the first chord runs from $0°$ clockwise to $-60°$; call this arc $X$. The second chord starts at a uniformly random angle $\theta$ and ends at $\theta - 60°$.

**Step 3: when do they cross?** Two chords cross exactly when one endpoint of the second chord lies inside arc $X$ and the other lies outside (their endpoints alternate around the circle).

- If $\theta$ is in $X$ (between $-60°$ and $0°$), its other end $\theta - 60°$ is past $-60°$, outside $X$: they cross.

- If $\theta$ is between $0^\circ$ and $60^\circ$, the start is outside $X$ but the end $\theta - 60^\circ$ falls inside $X$: they cross.
- For any other $\theta$, both ends lie outside $X$ (a $60^\circ$ arc cannot jump over $X$), so they do not cross.

**Step 4.** The good starting points fill $60^\circ + 60^\circ = 120^\circ$ of the $360^\circ$ circle, so the probability is $\dfrac{120}{360} = \dfrac{1}{3}$.

**Watch out.** It is easy to count only the case where the second chord *starts* on arc $X$, which gives $\frac{60}{360} = \frac{1}{6}$, choice (A). The chord that starts just before $X$ and swings into it crosses too.

**Answer: (D)** $\dfrac{1}{3}$