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# AMC 10

# SET 8

HALF-LENGTH PRACTICE

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# SOLUTIONS

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[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

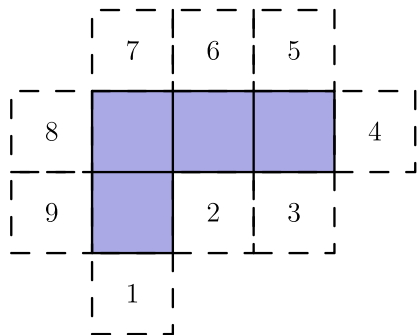
## Answer Key and Sources

| No. | Ans. | Source           | Topic                  |
|-----|------|------------------|------------------------|
| 1   | E    | 2003 AMC 10A #10 | Geometry               |
| 2   | B    | 2002 AMC 10A #11 | Algebra                |
| 3   | C    | 2019 AMC 10B #12 | Number Theory          |
| 4   | B    | 2012 AMC 10B #13 | Algebra                |
| 5   | E    | 2005 AMC 10A #14 | Counting & Probability |
| 6   | B    | 2012 AMC 10A #15 | Geometry               |
| 7   | B    | 2005 AMC 10B #17 | Algebra                |
| 8   | E    | 2011 AMC 10B #18 | Geometry               |
| 9   | E    | 2011 AMC 10A #19 | Number Theory          |
| 10  | C    | 2007 AMC 10B #20 | Counting & Probability |

1 Geometry

Source: 2003 AMC 10A #10

The polygon enclosed by the solid lines in the figure consists of 4 congruent squares joined edge-to-edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing?



- (A) 2                      (B) 3                      (C) 4                      (D) 5                      (E) 6

**Key idea.** Fold the four shaded squares first and name the cube faces they cover. An extra square works exactly when it folds onto one of the two faces that are still open.

Lay the left square of the top row flat as the *bottom*. The square to its right folds up as the *right* wall, and the square after that folds over as the *top*. The shaded square below the bottom folds up as the *front* wall. The open faces are the *left* wall and the *back* wall.

*Positions that work.* Squares 8 and 9 hang off the left edges of the bottom and the front wall, so both fold to the left wall. Squares 7, 6, 5 hang off the back edges of the bottom, the right wall and the top, so all three fold to the back wall. Square 4 continues past the top and folds down as the left wall.

*Positions that fail.* Squares 2 and 3 hang off the front edges of the right wall and the top, so they land on the front wall, which is taken. Square 1 continues past the front wall onto the top, also taken.

So positions 4 through 9 work: 6 polygons.

**Answer: (E) 6**

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**2** Algebra

Source: 2002 AMC 10A #11

Jamal wants to store 30 computer files on floppy disks, each of which has a capacity of 1.44 megabytes (mb). Three of his files require 0.8 mb of memory each, 12 more require 0.7 mb each, and the remaining 15 require 0.4 mb each. No file can be split between floppy disks. What is the minimal number of floppy disks that will hold all the files?

- (A) 12                      (B) 13                      (C) 14                      (D) 15                      (E) 16

**Key idea.** The total memory suggests 12 disks, but files cannot be split, so some space is always wasted. The awkward files are the big ones, so ask: what else can share a disk with a 0.8 mb file?

**Step 1.** The total memory is  $3(0.8) + 12(0.7) + 15(0.4) = 2.4 + 8.4 + 6.0 = 16.8$  mb.

**Step 2.** A 0.8 mb file cannot share with another 0.8 ( $1.6 > 1.44$ ), with a 0.7 ( $1.5 > 1.44$ ), or with two 0.4's ( $1.6 > 1.44$ ). So at most one 0.4 file joins it, and that disk holds at most 1.2 mb. The three 0.8 files must sit on three different disks.

**Step 3.** Suppose  $n$  disks are used. Three of them hold at most 1.2 mb and the rest at most 1.44 mb, so

$$16.8 \leq 3(1.2) + 1.44(n - 3), \quad \text{i.e.} \quad 13.2 \leq 1.44(n - 3).$$

Since  $13.2/1.44 \approx 9.17$ , we need  $n - 3 \geq 10$ , so  $n \geq 13$ .

**Step 4.** Thirteen disks suffice: 3 disks with  $0.8 + 0.4 = 1.2$ , 6 disks with  $0.7 + 0.7 = 1.4$ , and 4 disks with  $0.4 + 0.4 + 0.4 = 1.2$ . That uses all 3 big files, all 12 files of 0.7, and  $3 + 12 = 15$  small files.

**Watch out.** Dividing the total by the capacity gives  $16.8/1.44 \approx 11.67$ , which rounds up to 12, choice (A). That count pretends files can be split; the wasted space on the 0.8 disks pushes the answer to 13.

**Answer: (B) 13**

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**3** Number Theory

Source: 2019 AMC 10B #12

What is the greatest possible sum of the digits in the base-seven representation of a positive integer less than 2019?

- (A) 11                      (B) 14                      (C) 22                      (D) 23                      (E) 27

**Key idea.** A big digit sum needs lots of 6's, the largest base-seven digit. So write 2019 in base seven to see how many digits we have and how far each digit is allowed to go.

**Step 1.** Since  $7^3 = 343$  and  $7^4 = 2401 > 2019$ , the numbers have at most four base-seven digits. Dividing:  $2019 = 5 \cdot 343 + 304$ ,  $304 = 6 \cdot 49 + 10$ ,  $10 = 1 \cdot 7 + 3$ . So  $2019 = 5613_7$ . (Check:  $1715 + 294 + 7 + 3 = 2019$ .) Numbers with three or fewer digits have digit sum at most  $6 + 6 + 6 = 18$ .

**Step 2.** Leading digit at most 4: the sum is at most  $4 + 6 + 6 + 6 = 22$ , and  $4666_7 = 1714 < 2019$  reaches it.

**Step 3.** Leading digit 5: the number must be below  $5613_7$ .

- Second digit at most 5: sum at most  $5 + 5 + 6 + 6 = 22$  (reached by  $5566_7 = 2008$ ).
- Second digit 6: the third digit is 0 or 1. With 0, the sum is at most  $5 + 6 + 0 + 6 = 17$ ; with 1, the last digit is at most 2, so the sum is at most  $5 + 6 + 1 + 2 = 14$ .

No case beats 22, and 22 is achieved.

**Watch out.** The largest allowed number,  $2018 = 5612_7$ , has digit sum  $5 + 6 + 1 + 2 = 14$ , choice (B). A bigger number does not mean a bigger digit sum — lowering the leading digit to 4 to free up three 6's does better.

**Answer: (C) 22**

#### 4 Algebra

Source: 2012 AMC 10B #13

It takes Clea 60 seconds to walk down an escalator when it is not operating, and only 24 seconds to walk down the escalator when it is operating. How many seconds does it take Clea to ride down the operating escalator when she just stands on it?

- (A) 36                      (B) 40                      (C) 42                      (D) 48                      (E) 52

**Key idea.** Times do not add, but speeds do: walking on a moving escalator, Clea's speed is her walking speed *plus* the escalator's speed. Measure speed as “fraction of the escalator per second.”

Walking on the stopped escalator, Clea covers  $\frac{1}{60}$  of it per second. Walking on the moving escalator, she covers  $\frac{1}{24}$  of it per second.

The second rate is her walking rate plus the escalator's rate, so the escalator alone moves

$$\frac{1}{24} - \frac{1}{60} = \frac{5}{120} - \frac{2}{120} = \frac{3}{120} = \frac{1}{40}$$

of its length per second. Standing still, she therefore needs 40 seconds.

Check:  $\frac{1}{60} + \frac{1}{40} = \frac{2}{120} + \frac{3}{120} = \frac{5}{120} = \frac{1}{24}$ . ✓

**Watch out.** Subtracting the times,  $60 - 24 = 36$ , gives choice (A). Times cannot be subtracted like this; only rates combine by adding and subtracting.

**Another way.** Pretend the escalator has 120 steps (120 is a common multiple of 60 and 24). Walking alone, Clea covers  $120/60 = 2$  steps per second; on the moving escalator,  $120/24 = 5$  steps per second. So the escalator supplies  $5 - 2 = 3$  steps per second, and riding alone takes  $120/3 = 40$  seconds.

**Answer: (B) 40**

5 Counting & Probability

Source: 2005 AMC 10A #14

How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

- (A) 41                      (B) 42                      (C) 43                      (D) 44                      (E) 45

**Key idea.** If the middle digit is the average of the other two, it is completely determined by them. So count the possible (first, last) pairs — the only catch is that the average must be a whole number.

Write the number as  $\overline{abc}$  with  $a \in \{1, \dots, 9\}$  and  $c \in \{0, \dots, 9\}$ . We need  $b = \frac{a+c}{2}$ .

This is always between 0 and 9, so  $b$  is a valid digit exactly when  $a + c$  is even — that is, when  $a$  and  $c$  have the same parity. Each such pair gives exactly one number.

**Both odd.**  $a \in \{1, 3, 5, 7, 9\}$  and  $c \in \{1, 3, 5, 7, 9\}$ :  $5 \cdot 5 = 25$  pairs. Example: 741, since  $\frac{7+1}{2} = 4$ .

**Both even.**  $a \in \{2, 4, 6, 8\}$  (the leading digit cannot be 0) and  $c \in \{0, 2, 4, 6, 8\}$ :  $4 \cdot 5 = 20$  pairs. Example: 420.

Total:  $25 + 20 = 45$ .

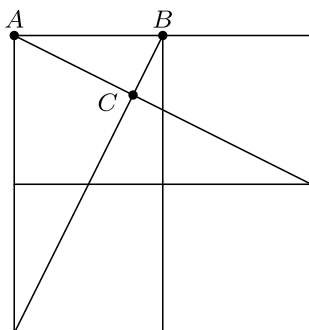
**Watch out.** Letting the first digit be 0 in the even case gives  $5 \cdot 5 = 25$  even pairs and a total of 50 — not among the choices, a sign the leading digit was mishandled.

**Answer: (E)** 45

6 Geometry

Source: 2012 AMC 10A #15

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of  $\triangle ABC$ ?



- (A)  $\frac{1}{6}$                       (B)  $\frac{1}{5}$                       (C)  $\frac{2}{9}$                       (D)  $\frac{1}{3}$                       (E)  $\frac{\sqrt{2}}{4}$

**Key idea.** The figure is built on a unit grid, so coordinates make each segment a line with a simple equation, and  $C$  is where they cross. Since  $\overline{AB}$  is a horizontal side of length 1, all we need is how far  $C$  lies below it.

**Step 1.** Put  $A = (0, 0)$  and  $B = (1, 0)$ , with the squares below the  $x$ -axis. The segment from  $A$  ends at the far corner  $(2, -1)$  of the right square, so it has slope  $-\frac{1}{2}$ :  $y = -\frac{1}{2}x$ . The segment from  $B$  ends at the bottom-left corner  $(0, -2)$  of the lower square, so its slope is  $\frac{-2-0}{0-1} = 2$ :  $y = 2x - 2$ .

**Step 2.** At  $C$ :  $2x - 2 = -\frac{1}{2}x$ , so  $\frac{5}{2}x = 2$ , giving  $x = \frac{4}{5}$  and  $y = -\frac{1}{2} \cdot \frac{4}{5} = -\frac{2}{5}$ .

**Step 3.** With base  $AB = 1$  and height  $\frac{2}{5}$ ,

$$[ABC] = \frac{1}{2} \cdot 1 \cdot \frac{2}{5} = \frac{1}{5}.$$

**Another way.** The slopes  $-\frac{1}{2}$  and 2 multiply to  $-1$ , so the segments are perpendicular and  $\angle ACB = 90^\circ$ . In right  $\triangle ABC$  the hypotenuse is  $AB = 1$ , and since the segment from  $A$  drops 1 for every 2 across, the angle at  $A$  satisfies  $\tan \angle CAB = \frac{BC}{AC} = \frac{1}{2}$ , so  $BC : AC = 1 : 2$ . Then  $AC^2 + BC^2 = 5BC^2 = 1$ , so  $BC = \frac{1}{\sqrt{5}}$ ,  $AC = \frac{2}{\sqrt{5}}$ , and the area is  $\frac{1}{2} \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \frac{1}{5}$ .

**Answer: (B)**  $\frac{1}{5}$

## 7 Algebra

Source: 2005 AMC 10B #17

Suppose that  $4^a = 5$ ,  $5^b = 6$ ,  $6^c = 7$ , and  $7^d = 8$ . What is  $a \cdot b \cdot c \cdot d$ ?

- (A) 1                      (B)  $\frac{3}{2}$                       (C) 2                      (D)  $\frac{5}{2}$                       (E) 3

**Key idea.** Each equation's answer is the next equation's base: 5 appears in  $4^a = 5$  and in  $5^b$ , and so on. That chain invites substitution, rewriting everything as a power of 4.

**Step 1.** Replace 5 by  $4^a$ :  $6 = 5^b = (4^a)^b = 4^{ab}$ .

Replace 6 by  $4^{ab}$ :  $7 = 6^c = 4^{abc}$ .

Replace 7 by  $4^{abc}$ :  $8 = 7^d = 4^{abcd}$ .

**Step 2.** Write both sides as powers of 2:  $4^{abcd} = 2^{2abcd}$  and  $8 = 2^3$ . So  $2abcd = 3$ , and

$$a \cdot b \cdot c \cdot d = \frac{3}{2}.$$

As a sanity check, each exponent is a little more than 1 (since  $4^1 = 4$  is just below 5, and so on):  $a \approx 1.161$ ,  $b \approx 1.113$ ,  $c \approx 1.086$ ,  $d \approx 1.069$ , and their product is about 1.5.

**Watch out.** Treating  $4^a = 5$  as if it said  $4a = 5$  gives  $a = \frac{5}{4}$ , and the product  $\frac{5}{4} \cdot \frac{6}{5} \cdot \frac{7}{6} \cdot \frac{8}{7}$  telescopes to 2, choice (C). But  $4^{5/4} \approx 5.66$ , not 5: exponents are not multipliers.

**Another way.** In logarithms,  $a = \frac{\log 5}{\log 4}$ ,  $b = \frac{\log 6}{\log 5}$ ,  $c = \frac{\log 7}{\log 6}$ ,  $d = \frac{\log 8}{\log 7}$ . The product telescopes to  $\frac{\log 8}{\log 4} = \frac{3 \log 2}{2 \log 2} = \frac{3}{2}$ .

**Answer: (B)**  $\frac{3}{2}$

## 8 Geometry

Source: 2011 AMC 10B #18

Rectangle  $ABCD$  has  $AB = 6$  and  $BC = 3$ . Point  $M$  is chosen on side  $AB$  so that  $\angle AMD = \angle CMD$ . What is the degree measure of  $\angle AMD$ ?

- (A) 15                      (B) 30                      (C) 45                      (D) 60                      (E) 75

**Key idea.** A rectangle has parallel sides, and parallel lines turn the given pair of equal angles into an isosceles triangle. Once we know  $MC$ , the numbers 6 and 3 point to a 30–60–90 triangle.

**Step 1.** Since  $\overline{AB} \parallel \overline{DC}$  with transversal  $\overline{MD}$ , the alternate angles are equal:  $\angle AMD = \angle MDC$ . Combined with  $\angle AMD = \angle CMD$ , this gives  $\angle MDC = \angle DMC$ .

**Step 2.** In  $\triangle MDC$  the angles at  $D$  and  $M$  are equal, so the sides opposite them are equal:  $MC = DC = AB = 6$ .

**Step 3.**  $\triangle MBC$  has a right angle at  $B$ , leg  $BC = 3$  and hypotenuse  $MC = 6$ . A leg equal to half the hypotenuse means  $\sin \angle CMB = \frac{3}{6} = \frac{1}{2}$ , so  $\angle CMB = 30^\circ$ . (Then  $MB = 3\sqrt{3} \approx 5.2 < 6$ , so  $M$  really is on side  $\overline{AB}$ .)

**Step 4.** Along the straight line  $AB$ ,  $\angle AMC = 180^\circ - 30^\circ = 150^\circ$ . Ray  $MD$  splits this angle into the two equal parts  $\angle AMD$  and  $\angle DMC$ , so

$$\angle AMD = \frac{1}{2} \cdot 150^\circ = 75^\circ.$$

**Watch out.** Guessing by symmetry that  $M$  is the midpoint of  $\overline{AB}$  gives  $\angle AMD = 45^\circ$ , choice (C) — but then  $\angle CMD = 90^\circ$ , so the condition fails.  $M$  is in fact much closer to  $A$ :  $AM = 6 - 3\sqrt{3} \approx 0.8$ .

**Answer: (E)** 75

## 9 Number Theory

Source: 2011 AMC 10A #19

In 1991 the population of a town was a perfect square. Ten years later, after an increase of 150 people, the population was 9 more than a perfect square. Now, in 2011, with an increase of another 150 people, the population is once again a perfect square. Which of the following is closest to the percent growth of the town's population during this twenty-year period?

- (A) 42                      (B) 47                      (C) 52                      (D) 57                      (E) 62

**Key idea.** Two perfect squares with a known gap between them call for a difference of squares:  $q^2 - p^2 = (q - p)(q + p)$ . Factoring one specific number leaves only a couple of cases to try.

**Step 1.** Let the 1991 population be  $p^2$  and write the 2001 population as  $q^2 + 9$ . Then  $p^2 + 150 = q^2 + 9$ , so  $q^2 - p^2 = 141$ .

**Step 2.** Factor:  $(q - p)(q + p) = 141 = 3 \cdot 47$ . Both factors are positive with  $q - p < q + p$ , so the pair is  $(1, 141)$  or  $(3, 47)$ .

- $q - p = 1$ ,  $q + p = 141$ : adding and halving gives  $q = 71$ ,  $p = 70$ .
- $q - p = 3$ ,  $q + p = 47$ : this gives  $q = 25$ ,  $p = 22$ .

**Step 3.** Now use 2011:  $p^2 + 300$  must be a perfect square. For  $p = 70$ :  $4900 + 300 = 5200$ , which lies strictly between  $72^2 = 5184$  and  $73^2 = 5329$  — not a square. For  $p = 22$ :  $484 + 300 = 784 = 28^2$ . ✓

So the populations were 484, then  $634 = 25^2 + 9$ , then 784.

**Step 4.** The growth over twenty years is

$$\frac{300}{484} \cdot 100\% \approx 62.0\%,$$

a little more than  $\frac{300}{500} = 60\%$ .

**Watch out.** Percent growth is measured from the *starting* population. Dividing by the 2001 population instead gives  $\frac{300}{634} \approx 47.3\%$ , choice (B).

**Answer: (E) 62**

**10** Counting & Probability

Source: 2007 AMC 10B #20

A set of 25 square blocks is arranged into a  $5 \times 5$  square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

- (A) 100                      (B) 125                      (C) 600                      (D) 2300                      (E) 3600

**Key idea.** “No two in the same row or column” is a condition on rows and on columns separately. So choose the rows, choose the columns, and then decide how to pair them up.

**Step 1.** The three blocks lie in three different rows:  $\binom{5}{3} = 10$  ways to pick which rows. Likewise  $\binom{5}{3} = 10$  ways to pick the columns.

**Step 2.** With the rows and columns fixed, each chosen row gets its own chosen column. The top row has 3 column choices, the next has 2, the last has 1:  $3! = 6$  pairings.

**Step 3.** Total:  $10 \cdot 10 \cdot 6 = 600$ .

**Another way.** Pick blocks one at a time. The first can be any of 25. Removing its row and column leaves a  $4 \times 4$  grid, so the second has 16 choices; similarly the third has 9. That is  $25 \cdot 16 \cdot 9 = 3600$  *ordered* picks, and each set of 3 blocks is counted  $3! = 6$  times, giving  $3600/6 = 600$ .

**Watch out.** Forgetting to divide out the order leaves 3600, choice (E). Forgetting the pairing step in the first method leaves  $10 \cdot 10 = 100$ , choice (A).

**Answer: (C) 600**