AMC 10

SET 9

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**D**	2004 AMC 10A #10	Counting & Probability
2	**C**	2001 AMC 10 #11	Algebra
3	**A**	2003 AMC 10A #12	Counting & Probability
4	**B**	2004 AMC 10B #13	Algebra
5	**C**	2016 AMC 10A #14	Number Theory
6	**E**	2011 AMC 10B #15	Algebra
7	**E**	2014 AMC 10A #16	Geometry
8	**B**	2010 AMC 10B #17	Algebra
9	**E**	2008 AMC 10B #19	Geometry
10	**B**	2013 AMC 10B #20	Number Theory

1 Counting & Probability

Source: 2004 AMC 10A #10

Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

(A) $\frac{19}{128}$ **(B)** $\frac{23}{128}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{35}{128}$ **(E)** $\frac{1}{2}$

Key idea. Coin A can show only 0, 1, 2 or 3 heads, so split into those four cases, count the ways for each coin, and multiply.

The number of ways to get k heads in n flips is $\binom{n}{k}$.

For coin A (3 flips, $2^3 = 8$ outcomes): $1,\ 3,\ 3,\ 1$ ways for $0, 1, 2, 3$ heads.

For coin B (4 flips, $2^4 = 16$ outcomes): $1,\ 4,\ 6,\ 4,\ 1$ ways for $0, 1, 2, 3, 4$ heads.

The two coins are independent, so there are $8 \cdot 16 = 128$ equally likely combined outcomes. The matching ones number

$$1 \cdot 1 + 3 \cdot 4 + 3 \cdot 6 + 1 \cdot 4 = 1 + 12 + 18 + 4 = 35.$$

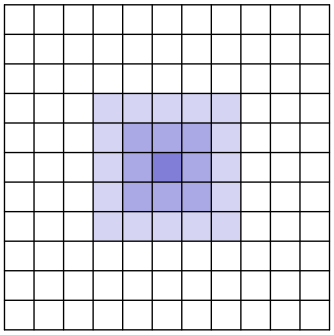
Four heads on coin B can never be matched, so that case adds nothing. The probability is $\frac{35}{128}$.

Answer: (D) $\frac{35}{128}$

2 Algebra

Source: 2001 AMC 10 #11

Consider the dark square in an array of unit squares, part of which is shown. The first ring of squares around this center square contains 8 unit squares. The second ring contains 16 unit squares. If we continue this process, the number of unit squares in the 100th ring is



(A) 396 **(B)** 404 **(C)** 800 **(D)** 10,000 **(E)** 10,404

Key idea. A ring is what is left when you remove a square from a slightly bigger square. The picture shows each ring as the outer layer of a filled square, which points straight at "big square minus small square".

Step 1. The center square together with the first ring is a 3×3 square. Adding the second ring gives a 5×5 square. Each ring adds one square on every side, so the side grows by 2 each time: the center plus the first n rings form a $(2n+1) \times (2n+1)$ square.

Step 2. The nth ring is therefore the $(2n+1) \times (2n+1)$ square with the $(2n-1) \times (2n-1)$ square inside it removed:

$$(2n+1)^2 - (2n-1)^2 = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = 8n.$$

Check: $n = 1$ gives 8 and $n = 2$ gives 16, matching the statement.

Step 3. For $n = 100$ the ring has $8 \cdot 100 = 800$ unit squares. Directly: $201^2 - 199^2 = 40401 - 39601 = 800$.

Watch out. Picturing the 100th ring as the border of a 100×100 square gives $4 \cdot 100 - 4 = 396$, choice (A). The 100th ring surrounds a 199×199 square, so it is the border of a 201×201 square.

Answer: (C) 800

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3 Counting & Probability

Source: 2003 AMC 10A #12

A point (x, y) is randomly picked from inside the rectangle with vertices $(0,0)$, $(4,0)$, $(4,1)$, and $(0,1)$. What is the probability that $x < y$?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{3}{4}$

Key idea. For a point chosen uniformly from a region, probability is a ratio of areas. The condition $x < y$ is a half-plane bounded by the line $y = x$, so sketch that line across the rectangle and measure the piece above it.

Step 1. The rectangle runs from $x = 0$ to $x = 4$ and from $y = 0$ to $y = 1$, so its area is $4 \cdot 1 = 4$.

Step 2. The points with $x < y$ lie above the line $y = x$. Inside the rectangle we always have $y \le 1$, so $x < y$ forces $x < 1$: only the left 1×1 square can contain such points.

In that unit square the line $y = x$ is the diagonal from $(0,0)$ to $(1,1)$. The part above it is the triangle with vertices $(0,0)$, $(0,1)$, $(1,1)$, a right triangle with legs 1 and 1, so its area is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$.

Step 3. The probability is

$$\frac{\text{favorable area}}{\text{total area}} = \frac{1/2}{4} = \frac{1}{8}.$$

Watch out. "$x < y$ and $x > y$ are symmetric, so the answer is $\frac{1}{2}$" leads to (D). That symmetry needs the region to be unchanged when x and y are swapped — true for a square, false for this 4×1 rectangle.

Answer: (A) $\frac{1}{8}$

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4 Algebra

Source: 2004 AMC 10B #13

> In the United States, coins have the following thicknesses: penny, 1.55 mm; nickel, 1.95 mm; dime, 1.35 mm; quarter, 1.75 mm. If a stack of these coins is exactly 14 mm high, how many coins are in the stack?
>
> **(A)** 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Key idea. The four thicknesses all end in 5 hundredths and differ from each other by multiples of 0.2 mm — a pattern like that is a hint to look at remainders. In hundredths of a millimetre each coin leaves remainder 15 when divided by 20, while the target 1400 leaves remainder 0; that restricts the number of coins, and rough size bounds finish the job.

Step 1: bound the count. Each coin is at most 1.95 mm and at least 1.35 mm thick. With k coins,

$$1.35k \le 14 \le 1.95k,$$

so $k \ge 14/1.95 \approx 7.2$ and $k \le 14/1.35 \approx 10.4$. Hence k is 8, 9, or 10.

Step 2: a divisibility condition. Measure in hundredths of a millimetre: the thicknesses are $135, 155, 175, 195$ and the stack is 1400. Each thickness is 15 more than a multiple of 20 ($135 = 120+15$, $155 = 140+15$, $175 = 160+15$, $195 = 180+15$). So the total of any k coins is (a multiple of 20) $+ 15k$.

Since 1400 is a multiple of 20, so is $15k$. Now $15k = 20m$ means $3k = 4m$; since 3 and 4 have no common factor, 4 must divide k.

Step 3. Of 8, 9, 10, only 8 is a multiple of 4. And 8 really works: eight quarters give $8 \cdot 1.75 = 14$ mm.

Answer: (B) 8

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5 Number Theory

Source: 2016 AMC 10A #14

> How many ways are there to write 2016 as the sum of twos and threes, ignoring order? (For example, $1008 \cdot 2 + 0 \cdot 3$ and $402 \cdot 2 + 404 \cdot 3$ are two such ways.)
>
> **(A)** 236 **(B)** 336 **(C)** 337 **(D)** 403 **(E)** 672

Key idea. "Ignoring order" means a way is fixed by *how many* twos and *how many* threes are used. So count the nonnegative integer solutions of $2x + 3y = 2016$ — and once the number of threes is chosen, the number of twos is forced.

Step 1. Let x be the number of twos and y the number of threes, so $2x + 3y = 2016$ with $x, y \ge 0$. For each y there is at most one x, namely $x = \dfrac{2016 - 3y}{2}$, and we need it to be a nonnegative integer.

Step 2. $2016-3y$ is even exactly when $3y$ is even, that is, when y is even. And $x \geq 0$ means $3y \leq 2016$, i.e. $y \leq 672$.

Step 3. So y can be any even number from 0 to 672:

$$y = 0, 2, 4, \ldots, 672,$$

which is $\dfrac{672}{2} + 1 = 337$ values. Each gives one way, from $1008 \cdot 2 + 0 \cdot 3$ (at $y = 0$) to $0 \cdot 2 + 672 \cdot 3$ (at $y = 672$).

Watch out. Computing $672 \div 2 = 336$ and stopping gives (B). The list $0, 2, \ldots, 672$ includes *both* endpoints, so there is one more value: 337.

Answer: (C) 337

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6 Algebra

Source: 2011 AMC 10B #15

Let @ denote the "averaged with" operation: $a@b = \frac{a+b}{2}$. Which of the following distributive laws hold for all numbers x, y, and z?

I. $x@(y+z) = (x@y) + (x@z)$

II. $x + (y@z) = (x+y)@(x+z)$

III. $x@(y@z) = (x@y)@(x@z)$

(A) I only **(B)** II only **(C)** III only

(D) I and III only **(E)** II and III only

Key idea. Each law is an identity, so just unfold every @ into $\frac{\cdot+\cdot}{2}$ and compare the two sides. If they differ, one numerical example proves the law fails.

Law I. Left side: $x@(y+z) = \dfrac{x+y+z}{2}$. Right side:

$$\frac{x+y}{2} + \frac{x+z}{2} = \frac{2x+y+z}{2} = x + \frac{y+z}{2}.$$

The right side has an extra $\frac{x}{2}$, so they differ whenever $x \neq 0$. For example, $x = 2$, $y = z = 0$: the left side is 1, the right side is $1 + 1 = 2$. Law I fails.

Law II. Left side: $x + \dfrac{y+z}{2}$. Right side:

$$(x+y)@(x+z) = \frac{(x+y)+(x+z)}{2} = \frac{2x+y+z}{2} = x + \frac{y+z}{2}.$$

Equal for all x, y, z, so Law II holds. (Shifting two numbers by x shifts their average by x.)

Law III. Left side: $x@\dfrac{y+z}{2} = \dfrac{x + \frac{y+z}{2}}{2} = \dfrac{2x+y+z}{4}$. Right side:

$$\frac{\frac{x+y}{2} + \frac{x+z}{2}}{2} = \frac{\frac{2x+y+z}{2}}{2} = \frac{2x+y+z}{4}.$$

Equal again, so Law III holds.

Only II and III are true.

Watch out. Law I looks the most like ordinary distributivity, $a(b+c) = ab + ac$, so it is tempting to accept it without checking — every choice containing I is wrong. Testing with $x = y = z = 0$ also misleads, since both sides are 0; pick an example with $x \neq 0$.

Answer: (E) II and III only

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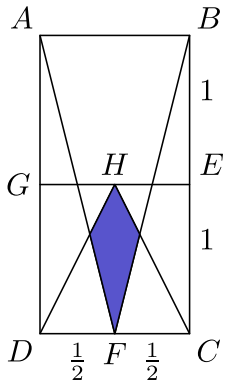
7 Geometry

Source: 2014 AMC 10A #16

In rectangle $ABCD$, $AB = 1$, $BC = 2$, and points E, F, and G are midpoints of $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Point H is the midpoint of $\overline{GE}$. What is the area of the shaded region?



(A) $\frac{1}{12}$ **(B)** $\frac{\sqrt{3}}{18}$ **(C)** $\frac{\sqrt{2}}{12}$ **(D)** $\frac{\sqrt{3}}{12}$ **(E)** $\frac{1}{6}$

Key idea. The shaded region is a kite: it is symmetric about the vertical line $\overline{HF}$. A kite's area is half the product of its diagonals, and one diagonal, $HF = 1$, is already known. The other needs only two lines intersected, which coordinates do cleanly.

Step 1: coordinates. Put $D = (0,0)$, $C = (1,0)$, $B = (1,2)$, $A = (0,2)$. Then $E = (1,1)$, $F = (\frac{1}{2},0)$, $G = (0,1)$, and H, the midpoint of $\overline{GE}$, is $(\frac{1}{2},1)$.

In the figure the shaded region has vertices H, F, the point P where $\overline{AF}$ meets $\overline{DH}$, and the point Q where $\overline{BF}$ meets $\overline{CH}$.

Step 2: find P. Line DH goes from $(0,0)$ to $(\frac{1}{2},1)$, so it is $y = 2x$. Line AF goes from $(0,2)$ to $(\frac{1}{2},0)$, so it is $y = 2 - 4x$. Setting $2x = 2 - 4x$ gives $x = \frac{1}{3}$, so $P = (\frac{1}{3},\frac{2}{3})$.

By the left–right symmetry of the figure about $x = \frac{1}{2}$, $Q = (\frac{2}{3},\frac{2}{3})$.

Step 3: area. The diagonals are HF (vertical, length 1) and PQ (horizontal, length $\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$). They are perpendicular, so

$$\text{Area} = \frac{1}{2} \cdot HF \cdot PQ = \frac{1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}.$$

Another way. Subtract instead of adding. Triangle DHC has base $DC = 1$ and height 1 (the height of H), so its area is $\frac{1}{2}$. Removing the kite from it leaves triangles DPF and FQC. Triangle DPF has base $DF = \frac{1}{2}$ and height $\frac{2}{3}$ (the height of P), so its area is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6}$, and FQC is its mirror image. The kite has area $\frac{1}{2} - 2 \cdot \frac{1}{6} = \frac{1}{6}$.

Answer: (E) $\frac{1}{6}$

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8 Algebra

Source: 2010 AMC 10B #17

> Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37^{th} and 64^{th}, respectively. How many schools are in the city?
>
> **(A)** 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

Key idea. Turn every fact into a condition on n, the number of schools. "Median" and "all scores different" together say the number of students is odd; "placed 64^{th}" and "highest on her team" give one lower and one upper bound.

Step 1: parity. There are $3n$ students. When the number of students is even, the median is the average of the two middle scores, and since all scores are different that average is nobody's score. Andrea's score *is* the median, so $3n$ is odd, hence n is odd, and Andrea placed exactly in the middle, at place $\frac{3n+1}{2}$ (counting 1^{st} as the highest score).

Step 2: lower bound. Carla placed 64^{th}, so there are at least 64 students: $3n \geq 64$. Since $3 \cdot 21 = 63$, this means $n \geq 22$, and as n is odd, $n \geq 23$.

Step 3: upper bound. Andrea had the highest score on her team, so she finished ahead of Beth:

$$\frac{3n+1}{2} < 37 \quad \Longrightarrow \quad 3n+1 < 74 \quad \Longrightarrow \quad 3n < 73,$$

so $n \leq 24$, and as n is odd, $n \leq 23$.

Step 4. Hence $n = 23$. Check: 69 students, Andrea is 35^{th}, ahead of Beth (37^{th}), and Carla's 64^{th} place fits within 69.

Watch out. Skipping the parity step leaves $64 \leq 3n < 73$, which allows $n = 22$, 23, and 24 — choices (A), (B), and (C) all survive. With 66 or 72 students the median is not any one student's score, so only the odd value works.

Answer: (B) 23

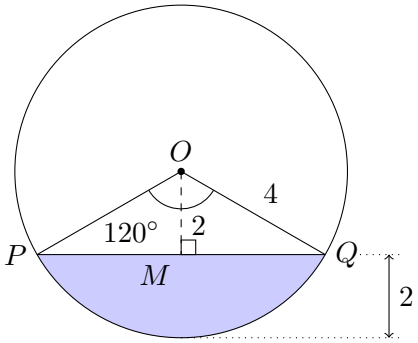
9 Geometry

Source: 2008 AMC 10B #19

A cylindrical tank with radius 4 feet and height 9 feet is lying on its side. The tank is filled with water to a depth of 2 feet. What is the volume of the water, in cubic feet?

(A) $24\pi - 36\sqrt{2}$ **(B)** $24\pi - 24\sqrt{3}$ **(C)** $36\pi - 36\sqrt{3}$ **(D)** $36\pi - 24\sqrt{2}$ **(E)** $48\pi - 36\sqrt{3}$

Key idea. With the tank on its side, every cross-section along its 9-foot length looks the same: a circle of radius 4 with water filling the piece below a chord. So the volume is (area of that piece) $\times 9$. A region cut off by a chord is found as *sector minus triangle*.



Step 1: the angle. The water is 2 feet deep and the radius is 4, so the surface $\overline{PQ}$ is $4-2=2$ feet below the center O. Let M be the foot of the perpendicular from O to $\overline{PQ}$; then $OM = 2$ and $OP = 4$. A right triangle whose hypotenuse is twice a leg is a 30–60–90 triangle, so $\angle POM = 60^\circ$ and $PM = 2\sqrt{3}$. Thus $\angle POQ = 120^\circ$ and $PQ = 4\sqrt{3}$.

Step 2: the cross-section. Sector POQ has area $\dfrac{120}{360}\cdot\pi\cdot 4^2 = \dfrac{16\pi}{3}$. Triangle POQ has area $\frac{1}{2}\cdot PQ\cdot OM = \frac{1}{2}\cdot 4\sqrt{3}\cdot 2 = 4\sqrt{3}$. The water's cross-section is the sector minus the triangle: $\dfrac{16\pi}{3} - 4\sqrt{3}$.

Step 3: the volume.

$$9\left(\frac{16\pi}{3} - 4\sqrt{3}\right) = 48\pi - 36\sqrt{3} \approx 88.4.$$

Sanity check: the full tank holds $\pi\cdot 4^2\cdot 9 = 144\pi \approx 452$ cubic feet, and a 2-foot layer at the bottom of an 8-foot-wide circle should be well under half of that — about a fifth is reasonable.

Answer: (E) $48\pi - 36\sqrt{3}$

10 Number Theory

Source: 2013 AMC 10B #20

The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!\cdots a_m!}{b_1!b_2!\cdots b_n!},$$

where

$$a_1 \geq a_2 \geq \cdots \geq a_m$$

and

$$b_1 \geq b_2 \geq \cdots \geq b_n$$

are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Key idea. Factorials and a specific number call for prime factorization. A prime p divides $k!$ exactly when $k \geq p$, so the largest prime factor of 2013 forces a_1 to be large — and that big factorial then drags in other primes the denominator has to cancel.

Step 1: a lower bound for a_1. $2013 = 3 \cdot 11 \cdot 61$, and 61 is prime. Clearing the fraction,

$$a_1!\,a_2!\cdots a_m! = 2013 \cdot b_1!\,b_2!\cdots b_n!,$$

so 61 divides the left side. A prime dividing a product divides one of the factors, so 61 divides some $a_i!$, which needs $a_i \geq 61$. Hence $a_1 \geq 61$.

Step 2: a lower bound for b_1. Now $a_1!$ includes the prime 59 as a factor (as $59 \leq 61$), so 59 divides the left side, hence the right side. But 59 does not divide $2013 = 3 \cdot 11 \cdot 61$, so 59 divides some $b_j!$, giving $b_1 \geq 59$. Therefore $a_1 + b_1 \geq 120$.

Step 3: 120 is achieved. Take $a_1 = 61$, $b_1 = 59$, so $\dfrac{61!}{59!} = 61 \cdot 60$. We still need to divide by 60 and multiply by $33 = 3 \cdot 11$, i.e. multiply by $\frac{11}{20}$: $\dfrac{11!}{10!} = 11$ and $\dfrac{3!}{5!} = \dfrac{1}{20}$. So

$$2013 = \frac{61! \cdot 11! \cdot 3!}{59! \cdot 10! \cdot 5!},$$

with $61 \geq 11 \geq 3$ and $59 \geq 10 \geq 5$. Check: $61 \cdot 60 \cdot 11 \cdot \frac{1}{20} = 61 \cdot 33 = 2013$.

Step 4. Since $a_1 \geq 61$ and $b_1 \geq 59$, the sum 120 forces $a_1 = 61$ and $b_1 = 59$ exactly. So $|a_1 - b_1| = 2$.

Watch out. The natural first try is $\frac{61!}{60!} = 61$, giving $2013 = \dfrac{61! \cdot 11! \cdot 3!}{60! \cdot 10! \cdot 2!}$ and $|a_1 - b_1| = 1$, choice (A). That representation is valid, but $a_1 + b_1 = 121$; dropping to 59! in the denominator does better.

Answer: (B) 2