AMC 10

SET 10

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**E**	2007 AMC 10A #10	Algebra
2	**E**	2003 AMC 10A #11	Number Theory
3	**D**	2016 AMC 10B #12	Counting & Probability
4	**E**	2015 AMC 10B #13	Geometry
5	**A**	2004 AMC 10A #14	Algebra
6	**D**	2018 AMC 10A #15	Geometry
7	**D**	2005 AMC 10A #16	Number Theory
8	**C**	2019 AMC 10B #17	Counting & Probability
9	**D**	2002 AMC 10B #18	Geometry
10	**D**	2013 AMC 10B #19	Algebra

1 Algebra

Source: 2007 AMC 10A #10

> The Dunbar family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?
>
> **(A)** 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Key idea. Turn each average into a *total*: total = average × number of people. Then write the family's total age in two ways.

Let N be the number of children, so the family has $N+2$ members and the "mother and children" group has $N+1$ members.

The whole family has average 20, so its total age is $20(N+2) = 20N+40$.

The same total is the father's 48 plus the mother-and-children total $16(N+1) = 16N+16$, which gives $16N+64$.

Setting the two totals equal: $20N+40 = 16N+64$, so $4N = 24$ and $N = 6$.

Watch out. The mother belongs to the group with average 16. Writing that group's total as $16N$ gives $20N+40 = 16N+48$, so $N = 2$ — choice (A).

Answer: (E) 6

2 Number Theory

Source: 2003 AMC 10A #11

> The sum of the two 5-digit numbers $AMC10$ and $AMC12$ is 123422. What is $A+M+C$?
>
> **(A)** 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Key idea. Both numbers start with the same three digits AMC, so treat AMC as a single 3-digit block. Writing each number as "block $\times 100$ plus its last two digits" turns the puzzle into one linear equation.

Step 1. Here $AMC10$ means the digits A, M, C, 1, 0 in a row, not a product. Shifting the block AMC two places left multiplies it by 100, so

$$AMC10 = 100 \cdot AMC + 10, \qquad AMC12 = 100 \cdot AMC + 12.$$

Step 2. Adding,

$$200 \cdot AMC + 22 = 123422,$$

so $200 \cdot AMC = 123400$ and $AMC = 123400 \div 200 = 617$.

Step 3. So $A = 6$, $M = 1$, $C = 7$. Check: $61710 + 61712 = 123422$. Hence $A+M+C = 6+1+7 = 14$.

Another way. Average instead of adding. The two numbers differ only in their last two digits, 10 and 12, so their average is $AMC11$. The average is also $123422 \div 2 = 61711$. Reading off the first three digits, $AMC = 617$, and $A + M + C = 14$.

Answer: (E) 14

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3 Counting & Probability

Source: 2016 AMC 10B #12

> Two different numbers are selected at random from $\{1, 2, 3, 4, 5\}$ and multiplied together. What is the probability that the product is even?
>
> **(A)** 0.2 **(B)** 0.4 **(C)** 0.5 **(D)** 0.7 **(E)** 0.8

Key idea. A product is even as soon as *one* factor is even, which splits into several cases. The opposite event is simple — the product is odd only when *both* numbers are odd — so count that and subtract from 1.

Step 1. The number of ways to choose two different numbers from five is $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$, all equally likely.

Step 2. The product is odd exactly when both numbers are odd. The odd numbers are 1, 3, 5, and the pairs from them are $\{1, 3\}$, $\{1, 5\}$, $\{3, 5\}$: that is $\binom{3}{2} = 3$ pairs.

Step 3. So the probability of an odd product is $\frac{3}{10} = 0.3$, and the probability of an even product is $1 - 0.3 = 0.7$.

Check by listing the even products directly: $\{1, 2\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{4, 5\}$ — seven of the ten pairs.

Watch out. Reasoning "two of the five numbers are even, so the chance is $\frac{2}{5} = 0.4$" (choice (B)) only checks one of the two numbers. The other number can supply the factor 2 just as well.

Answer: (D) 0.7

4 Geometry

Source: 2015 AMC 10B #13

The line $12x + 5y = 60$ forms a triangle with the coordinate axes. What is the sum of the lengths of the altitudes of this triangle?

(A) 20 **(B)** $\frac{360}{17}$ **(C)** $\frac{107}{5}$ **(D)** $\frac{43}{2}$ **(E)** $\frac{281}{13}$

Key idea. The coordinate axes meet at a right angle, so the triangle is a right triangle — and in a right triangle each leg is the altitude to the other leg. Only the altitude to the hypotenuse needs work, and the area gives it.

Step 1. Find where the line meets the axes. Putting $y = 0$ gives $12x = 60$, so $x = 5$; putting $x = 0$ gives $5y = 60$, so $y = 12$. The vertices are $(0, 0)$, $(5, 0)$ and $(0, 12)$.

Step 2. The legs have lengths 5 and 12, so the hypotenuse is $\sqrt{5^2 + 12^2} = \sqrt{169} = 13$. The leg of length 12 is perpendicular to the leg of length 5, so it is the altitude to that side, and likewise the other way round. Two altitudes are 12 and 5.

Step 3. The area is $\frac{1}{2} \cdot 5 \cdot 12 = 30$. If h is the altitude to the hypotenuse, the area is also $\frac{1}{2} \cdot 13 \cdot h$, so $13h = 60$ and $h = \frac{60}{13}$.

Step 4. The sum is

$$12 + 5 + \frac{60}{13} = \frac{156 + 65 + 60}{13} = \frac{281}{13}.$$

Watch out. Do not hunt for three new segments. Forgetting that the legs themselves are altitudes leaves only the $\frac{60}{13}$ you computed — not an answer choice.

Another way. The altitude to the hypotenuse is the distance from the origin to the line $12x + 5y - 60 = 0$, which is $\frac{|-60|}{\sqrt{12^2 + 5^2}} = \frac{60}{13}$.

Answer: (E) $\frac{281}{13}$

5 Algebra

Source: 2004 AMC 10A #14

> The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?
>
> **(A)** 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Key idea. Two averages with the same unknown count call for the usual move: turn each average into a total (average × number of coins). That pins down how many coins there are, and with so few coins the denominations can be sorted out by hand.

Step 1. Let n be the number of coins. Their total value is $20n$ cents. With one more quarter there are $n+1$ coins worth $20n+25$ cents, and the average is 21:

$$20n + 25 = 21(n+1) = 21n + 21,$$

so $n = 4$. Paula has 4 coins worth $20 \cdot 4 = 80$ cents.

Step 2. Count the quarters. Four quarters make 100 cents, too much. With at most two quarters, the other coins are worth at most 10 cents each, so the four coins total at most $2 \cdot 25 + 2 \cdot 10 = 70$ cents, too little. So there are exactly three quarters.

Step 3. Three quarters are 75 cents, leaving $80 - 75 = 5$ cents for the single remaining coin: a nickel. Her purse holds three quarters and one nickel ($25+25+25+5=80$), and no dimes.

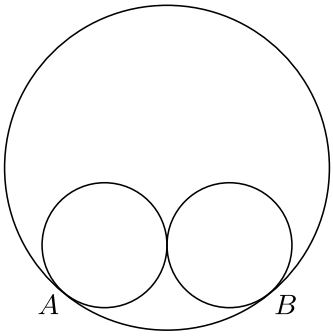
Watch out. The list "pennies, nickels, dimes, and quarters" names the kinds of coins allowed; it does not say she has one of each. Indeed one of each would be four coins worth only $1+5+10+25 = 41$ cents, not 80.

Answer: (A) 0

6 Geometry

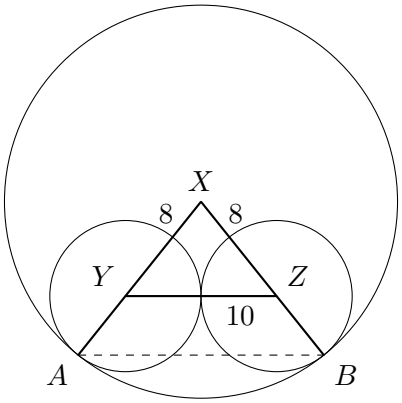
Source: 2018 AMC 10A #15

Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?



(A) 21 **(B)** 29 **(C)** 58 **(D)** 69 **(E)** 93

Key idea. Whenever circles are tangent, join their centers: the point of tangency lies on the line through the two centers. Here that puts A and B on rays from the big center, and $\overline{AB}$ becomes a scaled-up copy of the segment joining the small centers.



Step 1. Let X be the center of the big circle and Y, Z the centers of the small circles. The small circles are externally tangent, so $YZ = 5+5 = 10$. Each small circle is internally tangent to the big one, so $XY = XZ = 13-5 = 8$.

Step 2. The tangency point A lies on ray XY, at distance 13 from X (it is on the big circle). Likewise B lies on ray XZ with $XB = 13$.

Step 3. Triangles XAB and XYZ share the angle at X, and $\frac{XA}{XY} = \frac{XB}{XZ} = \frac{13}{8}$. So they are similar with ratio $\frac{13}{8}$, and

$$AB = \frac{13}{8}\cdot YZ = \frac{13}{8}\cdot 10 = \frac{130}{8} = \frac{65}{4}.$$

Since 65 and 4 are relatively prime, $m+n = 65+4 = 69$.

Watch out. Using the ratio upside down, $AB = \frac{8}{13} \cdot 10 = \frac{80}{13}$, gives $80 + 13 = 93$ — choice (E). A quick check catches it: A and B are farther from X than Y and Z, so AB must be *longer* than 10.

Answer: (D) 69

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7 Number Theory

Source: 2005 AMC 10A #16

The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 10 **(E)** 19

Key idea. Any question about the digits of a two-digit number invites writing it as $10a + b$. Here that pays off at once: the units digit b cancels when the digit sum is subtracted, so only the tens digit matters.

Step 1. Write the number as $10a + b$, where the tens digit a is 1 to 9 and the units digit b is 0 to 9. Subtracting the digit sum gives

$$(10a + b) - (a + b) = 9a.$$

For example, $47 - (4 + 7) = 47 - 11 = 36 = 9 \cdot 4$.

Step 2. The multiples $9a$ for $a = 1, 2, \ldots, 9$ are

$$9,\ 18,\ 27,\ 36,\ 45,\ 54,\ 63,\ 72,\ 81.$$

Only 36 has units digit 6, so $a = 4$.

Step 3. The result $9a$ does not depend on b at all, so every choice of b from 0 to 9 works. The numbers are $40, 41, \ldots, 49$ — ten of them.

Watch out. It is tempting to search for one special number and stop. Because b cancels, once $a = 4$ works, *all* ten values of b work.

Answer: (D) 10

8 Counting & Probability

Source: 2019 AMC 10B #17

A red ball and a green ball are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1, 2, 3, \ldots$. What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

(A) $\frac{1}{4}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{3}{7}$

Key idea. The two balls follow exactly the same rule, so "red higher" and "green higher" must be equally likely. That symmetry means you only need the probability of a *tie*, which is a simple geometric series.

Step 1. Exactly one of three things happens: red is higher, green is higher, or both land in the same bin. By symmetry the first two have the same probability p, so

$$2p + P(\text{same bin}) = 1.$$

Step 2. Both balls land in bin k with probability $2^{-k} \cdot 2^{-k} = 4^{-k}$. Adding over all bins,

$$P(\text{same bin}) = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1/4}{1 - 1/4} = \frac{1}{3}.$$

Step 3. Then $2p = 1 - \frac{1}{3} = \frac{2}{3}$, so $p = \frac{1}{3}$.

So all three outcomes — red higher, green higher, a tie — have probability exactly $\frac{1}{3}$.

Watch out. Saying "by symmetry the answer is $\frac{1}{2}$" forgets that the balls can share a bin; $\frac{1}{2}$ is not even a choice. Symmetry splits only what is left after the ties.

Another way. Directly: if red lands in bin r, green is lower with probability $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{r-1}} = 1 - 2^{1-r}$. So

$$p = \sum_{r=1}^{\infty} 2^{-r}\left(1 - 2^{1-r}\right) = \sum_{r=1}^{\infty} 2^{-r} - 2\sum_{r=1}^{\infty} 4^{-r} = 1 - \frac{2}{3} = \frac{1}{3}.$$

Answer: (C) $\frac{1}{3}$

9 Geometry

Source: 2002 AMC 10B #18

Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 16

Key idea. Every intersection point belongs to some *pair* of circles, and a pair can meet only so often. So bound the count pair by pair — then, because the question asks for a maximum, show a drawing that actually reaches the bound.

Step 1 (upper bound). Two distinct circles meet in at most 2 points. Four circles form $\binom{4}{2} = \frac{4\cdot 3}{2} = 6$ pairs. Each intersection point lies on at least two circles, so it is one of the meeting points of at least one pair. Hence there are at most $6 \cdot 2 = 12$ such points.

Step 2 (the bound is reached). Take four circles of radius 1 centered at the corners $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ of a unit square. The distance between any two centers is 1 or $\sqrt{2}$, both less than 2, so every pair of circles crosses in 2 points. No point lies on three of the circles, so the 12 points are all different. (For instance, the circles centered at $(0,0)$ and $(1,0)$ meet at $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, and neither point is at distance 1 from $(0,1)$ or $(1,1)$.)

So the maximum is 12.

Watch out. Counting "each circle meets the other three circles twice" gives $4 \cdot 3 \cdot 2 = 24$, but this counts every pair twice — once from each circle. Dividing by 2 brings it back to 12.

Answer: (D) 12

10 Algebra

Source: 2013 AMC 10B #19

The real numbers c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?

(A) $-7 - 4\sqrt{3}$ **(B)** $-2 - \sqrt{3}$ **(C)** -1 **(D)** $-2 + \sqrt{3}$ **(E)** $-7 + 4\sqrt{3}$

Key idea. Two signals: "arithmetic sequence" says write the terms around the middle one, $b - d$, b, $b + d$; "exactly one root" says the discriminant is zero. Together they give the ratio of d to b.

Step 1. Let the common difference be d, so $c = b - d$ and $a = b + d$. Since $a \geq b$, we have $d \geq 0$. Also $b > 0$: if $b = 0$ then $c \geq 0$ forces $d = 0$ and $a = 0$, which is not a quadratic.

Step 2. Exactly one root means $b^2 - 4ac = 0$:

$$b^2 = 4(b + d)(b - d) = 4b^2 - 4d^2,$$

so $4d^2 = 3b^2$ and, since $d \geq 0$, $d = \frac{\sqrt{3}}{2}\, b$.

Step 3. Only the ratios matter, so take $b = 2$. Then $d = \sqrt{3}$, $a = 2 + \sqrt{3}$ and $c = 2 - \sqrt{3} > 0$, so all the conditions hold. The double root is

$$-\frac{b}{2a} = -\frac{2}{2(2+\sqrt{3})} = -\frac{1}{2+\sqrt{3}} = -\frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = -(2-\sqrt{3}).$$

That is $-2 + \sqrt{3}$. Check the discriminant: $2^2 - 4(2+\sqrt{3})(2-\sqrt{3}) = 4 - 4(4-3) = 0$.

Watch out. Taking the negative square root, $d = -\sqrt{3}$, ignores $a \geq b$. It gives $a = 2 - \sqrt{3}$ and the root $-\frac{1}{2-\sqrt{3}} = -2 - \sqrt{3}$ — choice (B).

Answer: (D) $-2 + \sqrt{3}$