
AMC 10

SET 11

HALF-LENGTH PRACTICE

SOLUTIONS

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All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

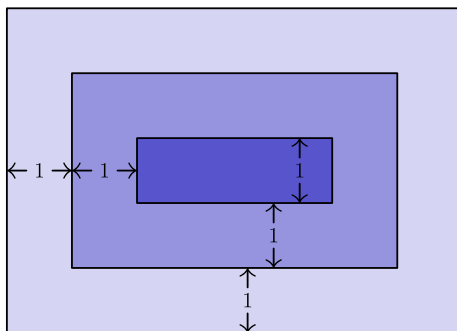
Answer Key and Sources

No.	Ans.	Source	Topic
1	B	2016 AMC 10A #10	Algebra
2	B	2005 AMC 10A #11	Geometry
3	E	2003 AMC 10B #13	Number Theory
4	A	2001 AMC 10 #14	Algebra
5	C	2017 AMC 10A #15	Counting & Probability
6	B	2015 AMC 10A #16	Algebra
7	D	2006 AMC 10B #17	Counting & Probability
8	E	2014 AMC 10B #18	Algebra
9	D	2000 AMC 10 #19	Geometry
10	D	2014 AMC 10A #20	Number Theory

1 Algebra

Source: 2016 AMC 10A #10

A rug is made with three different colors as shown. The areas of the three differently colored regions form an arithmetic progression. The inner rectangle is one foot wide, and each of the two shaded regions is 1 foot wide on all four sides. What is the length in feet of the inner rectangle?



- (A) 1 (B) 2 (C) 4 (D) 6 (E) 8

Key idea. Call the unknown length ℓ , write each colored area in terms of ℓ , and make the two differences equal.

Step 1. The inner rectangle is 1 by ℓ , so its area is ℓ .

Step 2. Adding a 1-foot band on every side adds 2 to each dimension, so the middle rectangle is 3 by $\ell + 2$. The middle *region* is that rectangle minus the inner one: $3(\ell + 2) - \ell = 2\ell + 6$.

Step 3. The whole rug is 5 by $\ell + 4$. The outer region is $5(\ell + 4) - 3(\ell + 2) = 2\ell + 14$.

Step 4. In an arithmetic progression the gaps are equal. The second gap is $(2\ell + 14) - (2\ell + 6) = 8$, so the first gap $(2\ell + 6) - \ell = \ell + 6$ must also be 8. Hence $\ell = 2$.

Check: the areas are 2, 10, 18, which go up by 8 each time.

Watch out. The middle and outer regions are frames. Using the whole rectangles $3(\ell + 2)$ and $5(\ell + 4)$ as the areas gives areas ℓ , $3\ell + 6$, $5\ell + 20$, whose gaps $2\ell + 6$ and $2\ell + 14$ can never be equal — no solution at all.

Answer: (B) 2

2 Geometry

Source: 2005 AMC 10A #11

A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n ?

- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Key idea. Count two things separately — all the small faces, and the red ones — and set their ratio to $\frac{1}{4}$. The red faces are exactly the unit squares that made up the big cube's painted surface, so they are easy to count without thinking about which cubes are corners or edges.

Step 1. There are n^3 unit cubes and each has 6 faces, so there are $6n^3$ small faces in total.

Step 2. Only the outside of the big cube was painted. Each of its 6 faces is an $n \times n$ grid of unit squares, and each of those squares is one face of one unit cube. So exactly $6n^2$ small faces are red. Faces created by the cutting are all unpainted.

Step 3. The fraction of red faces is

$$\frac{\text{red faces}}{\text{all faces}} = \frac{6n^2}{6n^3} = \frac{1}{n}.$$

Setting $\frac{1}{n} = \frac{1}{4}$ gives $n = 4$.

Check: with $n = 4$ there are 64 unit cubes with 384 faces, and $6 \cdot 16 = 96$ of them are red; $\frac{96}{384} = \frac{1}{4}$.

Watch out. The question is about *faces*, not cubes. Counting the unit cubes that have some red paint ($n^3 - (n-2)^3$ of them) answers a different question: for $n = 4$ that is 56 of the 64 cubes, nowhere near one-fourth.

Answer: (B) 4

3 Number Theory

Source: 2003 AMC 10B #13

Let $\clubsuit(x)$ denote the sum of the digits of the positive integer x . For example, $\clubsuit(8) = 8$ and $\clubsuit(123) = 1 + 2 + 3 = 6$. For how many two-digit values of x is $\clubsuit(\clubsuit(x)) = 3$?

- (A) 3 (B) 4 (C) 6 (D) 9 (E) 10

Key idea. Work from the outside in. Call $y = \clubsuit(x)$; the condition only says $\clubsuit(y) = 3$. Because x has just two digits, y is small, so there are very few possible values of y — find those first, then count the x that produce each.

Step 1. How big can y be? A two-digit number has digits at most 9 and 9, so $y = \clubsuit(x) \leq 18$. Also $y \geq 1$.

Step 2. Which y from 1 to 18 have digit sum 3? Only $y = 3$ and $y = 12$. (The next one, 21, is larger than 18.)

Step 3. Two-digit x with digit sum 3: the tens digit can be 1, 2 or 3, giving 12, 21, 30. That is 3 numbers.

Step 4. Two-digit x with digit sum 12: the tens digit a and units digit $12 - a$ must both be at most 9, so a runs from 3 to 9:

$$39, 48, 57, 66, 75, 84, 93.$$

That is 7 numbers.

Step 5. In total there are $3 + 7 = 10$ values of x .

Watch out. Stopping at “the digit sum of x is 3” forgets that $\clubsuit$ is applied twice; that counts only 12, 21, 30 and gives 3, choice (A). Forgetting 30 (a units digit of 0 is allowed) leaves a total of $2 + 7 = 9$, choice (D).

Answer: (E) 10

4 Algebra

Source: 2001 AMC 10 #14

A charity sells 140 benefit tickets for a total of \$2001. Some tickets sell for full price (a whole dollar amount), and the rest sell for half price. How much money is raised by the full-price tickets?

- (A) \$782 (B) \$986 (C) \$1158 (D) \$1219 (E) \$1449

Key idea. There are two unknowns (how many full-price tickets, and the price) but only one equation. What rescues us is that both unknowns are whole numbers: rearrange the equation into “product = number” form and look at the factors of that number.

Step 1. Let n tickets sell at the full price of p dollars, so $140 - n$ tickets sell at $\frac{p}{2}$ dollars. Then

$$np + (140 - n) \cdot \frac{p}{2} = 2001.$$

Step 2. Multiply by 2 to clear the fraction: $2np + 140p - np = 4002$, that is,

$$p(n + 140) = 4002.$$

Step 3. Factor: $4002 = 2 \cdot 3 \cdot 23 \cdot 29$. Since “some” tickets are full price and the rest half price, $1 \leq n \leq 139$, so $n + 140$ is a divisor of 4002 between 141 and 279. The divisors of 4002 near that range are 138, 174 and 667; only $174 = 2 \cdot 3 \cdot 29$ fits.

Step 4. So $n + 140 = 174$, giving $n = 34$ full-price tickets and $p = \frac{4002}{174} = 23$ dollars. The full-price tickets raise $34 \cdot 23 = \$782$.

Check: the other 106 tickets sell at \$11.50, raising \$1219, and $782 + 1219 = 2001$.

Watch out. Read the question carefully: it asks for the money from the *full-price* tickets. The half-price tickets raise $106 \cdot \$11.50 = \1219 , which is choice (D).

Answer: (A) \$782

5 Counting & Probability

Source: 2017 AMC 10A #15

Chloé chooses a real number uniformly at random from the interval $[0, 2017]$.

Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$.

What is the probability that Laurent's number is greater than Chloé's number?

- (A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) $\frac{5}{6}$ (E) $\frac{7}{8}$

Key idea. Laurent's interval is exactly twice Chloé's, so split on which half of $[0, 4034]$ Laurent lands in. In the top half he wins for sure; in the bottom half both pick from the same interval, so symmetry gives a fair coin.

Step 1. With probability $\frac{1}{2}$ Laurent's number lies in $(2017, 4034]$. Chloé's number is at most 2017, so Laurent's is greater for certain.

Step 2. With probability $\frac{1}{2}$ Laurent's number lies in $[0, 2017]$, and then it is uniform on the same interval as Chloé's. Neither player has an advantage, and a tie has probability 0 for real numbers, so Laurent is greater with probability $\frac{1}{2}$.

Step 3. Combine the two cases:

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

Another way. Picture the pair (Chloé, Laurent) as a point chosen uniformly in a 2017×4034 rectangle, with Chloé's number on the horizontal axis. Laurent's number is *not* greater exactly when the point is on or below the line "Laurent = Chloé", which cuts off a right triangle with legs 2017 and 2017. Its area is $\frac{1}{2} \cdot 2017^2$, a fraction

$$\frac{\frac{1}{2} \cdot 2017^2}{2017 \cdot 4034} = \frac{1}{4}$$

of the rectangle. So the answer is $1 - \frac{1}{4} = \frac{3}{4}$.

Watch out. "Two random numbers, each equally likely to be bigger" is only true when they come from the same interval. Using that here gives $\frac{1}{2}$, choice (A).

Answer: (C) $\frac{3}{4}$

6 Algebra

Source: 2015 AMC 10A #16

If we have that

$$y + 4 = (x - 2)^2,$$

$$x + 4 = (y - 2)^2,$$

and $x \neq y$, what is the value of

$$x^2 + y^2?$$

(A) 10

(B) 15

(C) 20

(D) 25

(E) 30

Key idea. Swapping x and y turns one equation into the other, so the natural tools are to *subtract* them and to *add* them. Subtracting produces a factor $x - y$, and the condition $x \neq y$ is exactly what lets you cancel it.

Step 1. Subtract the second equation from the first:

$$y - x = (x - 2)^2 - (y - 2)^2.$$

The right side is a difference of squares: $((x - 2) - (y - 2))((x - 2) + (y - 2)) = (x - y)(x + y - 4)$. The left side is $-(x - y)$.

Step 2. So $-(x - y) = (x - y)(x + y - 4)$. Because $x \neq y$, we may divide by $x - y$:

$$-1 = x + y - 4, \quad \text{so} \quad x + y = 3.$$

Step 3. Now add the two equations:

$$x + y + 8 = (x - 2)^2 + (y - 2)^2 = x^2 + y^2 - 4(x + y) + 8.$$

Cancel the 8s and move $-4(x + y)$ across: $x^2 + y^2 = 5(x + y) = 5 \cdot 3 = 15$.

Check: the solutions with $x \neq y$ are $x, y = \frac{3 \pm \sqrt{21}}{2}$ (in either order); their sum is 3 and their squares add to $\frac{(30+6\sqrt{21})+(30-6\sqrt{21})}{4} = 15$.

Why the condition matters: if $x = y$, both equations become $x + 4 = (x - 2)^2$, with solutions $x = y = 0$ and $x = y = 5$, giving $x^2 + y^2 = 0$ or 50. The problem rules these out.

Watch out. After finding $x + y = 3$, do not square it and stop: $(x + y)^2 = 9$ is not $x^2 + y^2$, because it also contains $2xy$.

Answer: (B) 15

7 Counting & Probability

Source: 2006 AMC 10B #17

Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?

- (A) $\frac{1}{10}$ (B) $\frac{1}{6}$ (C) $\frac{1}{5}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

Key idea. Do not track which *ball* goes where — track *colors*. Whatever color Alice sends, the bags end up equal exactly when Bob sends back that same color, so the problem collapses to one draw from Bob's bag.

Step 1. Say Alice moves a ball of color X . Now Alice has the four colors other than X , and Bob has six balls: one of each color, plus a second ball of color X .

Step 2. Bob returns a ball of color Y . If $Y = X$, Alice gets back her missing color and has one of each color; Bob is left with one of each color too. The bags match. If $Y \neq X$, Alice now has two balls of color Y and none of color X , while Bob still has two of color X . The bags differ.

Step 3. So we need Bob to pick one of the two balls of color X out of his six balls. This does not depend on which color X was, so the probability is

$$\frac{2}{6} = \frac{1}{3}.$$

Watch out. It is tempting to say Bob must return “the ball Alice gave him”, which happens with probability $\frac{1}{6}$, choice (B). But Bob's own ball of color X works just as well — the bags only care about colors.

Answer: (D) $\frac{1}{3}$

8 Algebra

Source: 2014 AMC 10B #18

A list of 11 positive integers has a mean of 10, a median of 9, and a unique mode of 8. What is the largest possible value of an integer in the list?

- (A) 24 (B) 30 (C) 31 (D) 33 (E) 35

Key idea. The total is fixed at $11 \cdot 10 = 110$, so making the largest number big is the same as making the other ten numbers *small*. The real question is how many 8s to use: more 8s let other values repeat more (cheap), but 8s also occupy slots where 1s could go. Try each count.

Step 1. Write the list in increasing order $a_1 \leq a_2 \leq \cdots \leq a_{11}$. The median is $a_6 = 9$, and the largest value is $a_{11} = 110 - (a_1 + \cdots + a_{10})$. Since $8 < 9$, every 8 sits among $a_1, \dots, a_5$, so there are at most

five 8s. There must be at least two, or no value would repeat and 8 would not be the unique mode. If 8 appears k times, every other value may appear at most $k - 1$ times.

Step 2. Find the smallest possible $a_1 + \cdots + a_{10}$ for each k :

- $k = 2$: other values at most once. Smallest: 1, 2, 3, 8, 8, 9, 10, 11, 12, 13, sum 77, so $a_{11} \leq 33$.
- $k = 3$: other values at most twice. Smallest: 1, 1, 8, 8, 8, 9, 9, 10, 10, 11, sum 75, so $a_{11} \leq 35$.
- $k = 4$: smallest 1, 8, 8, 8, 8, 9, 9, 9, 10, 10, sum 80, so $a_{11} \leq 30$.
- $k = 5$: smallest 8, 8, 8, 8, 8, 9, 9, 9, 9, 10, sum 86, so $a_{11} \leq 24$.

Each list is built greedily: below a_6 use the smallest values as often as allowed, then from $a_6 = 9$ onward use 9 as often as allowed (counting a_6 itself), then 10, and so on.

Step 3. The best bound is 35, and it is achieved: the list

$$1, 1, 8, 8, 8, 9, 9, 10, 10, 11, 35$$

has sum 110 and median 9; 8 appears three times and every other value at most twice.

Watch out. Most students use just two 8s, since that seems the “cheapest” mode. That forces every other value to be distinct and gives 33, choice (D). Allowing three 8s lets 1, 9 and 10 repeat, saving two more.

Answer: (E) 35

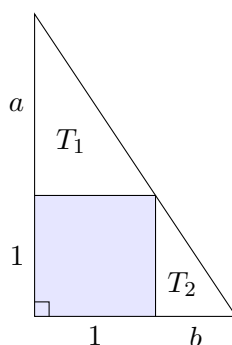
9 Geometry

Source: 2000 AMC 10 #19

Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. The ratio of the area of the other small right triangle to the area of the square is

- (A) $\frac{1}{2m+1}$ (B) m (C) $1-m$ (D) $\frac{1}{4m}$ (E) $\frac{1}{8m^2}$

Key idea. Only ratios matter, so let the square have side 1. The two small triangles have parallel sides, so they are similar — and similarity ties the leg of one to the leg of the other.



Step 1. The top triangle T_1 has a horizontal leg 1 (the top of the square) and a vertical leg a . The side triangle T_2 has a vertical leg 1 (the side of the square) and a horizontal leg b .

Step 2. Each small triangle's sides are parallel to the big triangle's sides, so T_1 and T_2 are similar, with vertical legs matching vertical legs. Comparing vertical leg to horizontal leg,

$$\frac{a}{1} = \frac{1}{b}, \quad \text{so} \quad ab = 1.$$

Step 3. The square has area 1. The two small triangles play symmetric roles, so we may call T_1 the one whose area is m times the square's. Then $\frac{1}{2} \cdot a \cdot 1 = m$, so $a = 2m$. Hence $b = \frac{1}{a} = \frac{1}{2m}$, and the area of T_2 is

$$\frac{1}{2} \cdot b \cdot 1 = \frac{1}{4m}.$$

Dividing by the square's area 1 gives the ratio $\frac{1}{4m}$.

Watch out. Testing a special case is a good idea, but not the isosceles one: there $m = \frac{1}{2}$ and *every* choice equals $\frac{1}{2}$, so nothing is eliminated. Try $a = 2$ instead: then $m = 1$ and $b = \frac{1}{2}$, so the other ratio is $\frac{1}{4}$, and only (D) gives $\frac{1}{4}$ at $m = 1$.

Answer: (D) $\frac{1}{4m}$

10 Number Theory

Source: 2014 AMC 10A #20

The product $(8)(888\dots 8)$, where the second factor has k digits, is an integer whose digits have a sum of 1000. What is k ?

- (A) 901 (B) 911 (C) 919 (D) 991 (E) 999

Key idea. With k near 1000 you cannot compute the product, but you can compute it for $k = 1, 2, 3, 4$ and spot the pattern. Then do the multiplication by hand, column by column, to see *why* the pattern continues — the carry settles into a repeating value.

Step 1. Small cases:

$$8 \cdot 8 = 64, \quad 8 \cdot 88 = 704, \quad 8 \cdot 888 = 7104, \quad 8 \cdot 8888 = 71104.$$

It looks like the product is 7, then a string of 1s, then 04.

Step 2. Why: multiply $8 \cdot 888\dots 8$ from the right, one digit at a time.

- Units column: $8 \cdot 8 = 64$. Write 4, carry 6.
- Tens column: again $8 \cdot 8 = 64$, plus the carry: $64 + 6 = 70$. Write 0, carry 7.
- Every later column: $8 \cdot 8$ plus the carry is $64 + 7 = 71$. Write 1, carry 7 again.
- After the last (k -th) column, the leftover carry 7 is written in front.

The columns after the first two are columns 3 through k , so there are $k - 2$ ones (for $k \geq 2$).

Step 3. The digit sum is

$$7 + \underbrace{1 + 1 + \dots + 1}_{k-2} + 0 + 4 = k + 9.$$

Setting $k + 9 = 1000$ gives $k = 991$.

Check: for $k = 4$ the formula gives 13, and $7 + 1 + 1 + 0 + 4 = 13$.

Answer: (D) 991