
AMC 10

SET 12

HALF-LENGTH PRACTICE

SOLUTIONS

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All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	C	2019 AMC 10A #10	Counting & Probability
2	A	2010 AMC 10B #11	Algebra
3	C	2014 AMC 10B #12	Number Theory
4	D	2002 AMC 10B #13	Algebra
5	B	2007 AMC 10A #15	Geometry
6	E	2003 AMC 10B #16	Counting & Probability
7	C	2004 AMC 10A #17	Algebra
8	D	2002 AMC 10A #18	Geometry
9	E	2018 AMC 10B #19	Number Theory
10	E	2019 AMC 10B #20	Geometry

1 Counting & Probability

Source: 2019 AMC 10A #10

A rectangular floor that is 10 feet wide and 17 feet long is tiled with 170 one-foot square tiles. A bug walks from one corner to the opposite corner in a straight line. Including the first and the last tile, how many tiles does the bug visit?

- (A) 17 (B) 25 (C) 26 (D) 27 (E) 28

Key idea. The bug enters a new tile exactly when it crosses a grid line, so count the grid lines it crosses and add the tile it starts on.

The floor is 17 tiles long and 10 tiles wide. Inside the floor there are $17 - 1 = 16$ grid lines running across the length and $10 - 1 = 9$ grid lines running along it. The straight path from corner to corner crosses each of these lines exactly once.

Could the bug cross two lines at the same moment? Only by passing through a corner where four tiles meet. That would need a point (x, y) with whole-number coordinates and $\frac{y}{x} = \frac{10}{17}$ strictly between the ends — impossible, because 10 and 17 have no common factor. So every crossing opens exactly one new tile.

Tiles visited: $1 + 16 + 9 = 26$.

Watch out. Simply adding $10 + 17 = 27$ (choice D) overcounts by one: there are 16 and 9 *interior* lines, not 17 and 10, and you must add 1 for the starting tile.

Answer: (C) 26**2** Algebra

Source: 2010 AMC 10B #11

A shopper plans to purchase an item that has a listed price greater than \$100 and can use any one of the three coupons. Coupon A gives 15% off the listed price, Coupon B gives \$30 off the listed price, and Coupon C gives 25% off the amount by which the listed price exceeds \$100.

Let x and y be the smallest and largest prices, respectively, for which Coupon A saves at least as many dollars as Coupon B or C. What is $y - x$?

- (A) 50 (B) 60 (C) 75 (D) 80 (E) 100

Key idea. Write each coupon's saving as a formula in the price p and compare A with each rival separately. The question asks for a *smallest* and a *largest* price, which is a hint that one comparison gives the lower end and the other gives the upper end.

Step 1. The savings in dollars at listed price p are

$$\text{A: } 0.15p, \quad \text{B: } 30, \quad \text{C: } 0.25(p - 100) = 0.25p - 25.$$

Step 2. A against B: $0.15p \geq 30$ means $p \geq \frac{30}{0.15} = 200$. A's saving grows with the price while B's is fixed, so this is the lower end.

Step 3. A against C: $0.15p \geq 0.25p - 25$ means $25 \geq 0.10p$, so $p \leq 250$. C gains 25 cents per extra dollar of price while A gains only 15, so C eventually overtakes A; this is the upper end.

Step 4. Both comparisons hold exactly when $200 \leq p \leq 250$, so $x = 200$, $y = 250$ and $y - x = 50$.

Check the ends: at $p = 200$ the savings are \$30, \$30, \$25; at $p = 250$ they are \$37.50, \$30, \$37.50. In both cases A is at least as good as each of the others.

Watch out. Coupon A has to keep up with B *and* with C. If you compare A only with B, A's saving keeps growing above \$30 forever and there is no largest price at all.

Answer: (A) 50

3 Number Theory

Source: 2014 AMC 10B #12

The largest divisor of 2,014,000,000 is itself. What is its fifth-largest divisor?

- (A) 125,875,000 (B) 201,400,000 (C) 251,750,000 (D) 402,800,000 (E) 503,500,000

Key idea. Divisors come in pairs d and $\frac{N}{d}$, so the fifth-*largest* divisor is N divided by the fifth-*smallest*. The statement itself hints at this: “the largest divisor is itself” is just $N \div 1$. Small divisors are easy to list; large ones are not.

Step 1. Factor the number. Write $N = 2,014,000,000 = 2014 \times 10^6$. Since $2014 = 2 \cdot 19 \cdot 53$ and $10^6 = 2^6 \cdot 5^6$,

$$N = 2^7 \cdot 5^6 \cdot 19 \cdot 53.$$

Step 2. List the smallest divisors. The only primes dividing N are 2, 5, 19, 53, so 3, 6 and 7 are not divisors. Going up from 1:

$$1, 2, 4, 5, 8, 10, \dots$$

The fifth-smallest divisor is 8.

Step 3. As d increases through the divisors, $\frac{N}{d}$ decreases through them. So the five largest divisors are $\frac{N}{1}, \frac{N}{2}, \frac{N}{4}, \frac{N}{5}, \frac{N}{8}$, and the fifth-largest is

$$\frac{2,014,000,000}{8} = 251,750,000.$$

Watch out. It is easy to list powers of 2 and forget 5. Using 1, 2, 4, 8, 16 gives $\frac{N}{16} = 125,875,000$, which is choice (A).

Answer: (C) 251,750,000

4 Algebra

Source: 2002 AMC 10B #13

What value of x makes $8xy - 12y + 2x - 3 = 0$ true for all values of y ?

- (A) $\frac{2}{3}$ (B) $\frac{3}{2}$ or $-\frac{1}{4}$ (C) $-\frac{2}{3}$ or $-\frac{1}{4}$ (D) $\frac{3}{2}$ (E) $-\frac{3}{2}$ or $-\frac{1}{4}$

Key idea. “True for all values of y ” means y must not matter. Group the terms that contain y and factor; the factor that does not involve y is the one that has to be zero.

Step 1. Group the terms in pairs and factor:

$$8xy - 12y + 2x - 3 = 4y(2x - 3) + (2x - 3) = (4y + 1)(2x - 3).$$

Step 2. The product must be 0 for *every* y . Try $y = 0$: the first factor is $4 \cdot 0 + 1 = 1$, so we need $2x - 3 = 0$, that is, $x = \frac{3}{2}$.

Step 3. Conversely, if $x = \frac{3}{2}$ then the second factor is 0, so the product is 0 no matter what y is. Directly: $8 \cdot \frac{3}{2}y - 12y + 2 \cdot \frac{3}{2} - 3 = 12y - 12y + 3 - 3 = 0$.

So the only value is $x = \frac{3}{2}$.

Another way. Think of the left side as a line in y : $(8x - 12)y + (2x - 3)$. It is zero for all y only if the slope and the constant are both zero: $8x - 12 = 0$ and $2x - 3 = 0$. Both give $x = \frac{3}{2}$.

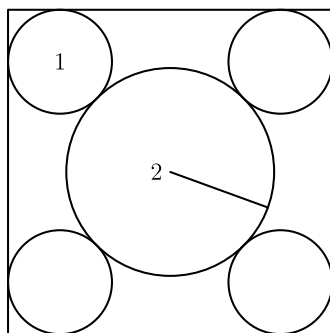
Watch out. After factoring, it is tempting to “solve” $(4y + 1)(2x - 3) = 0$ as an ordinary equation and write $x = \frac{3}{2}$ or $-\frac{1}{4}$ (choice B). But $-\frac{1}{4}$ is a value of y , not of x , and it only makes the expression zero for that one y .

Answer: (D) $\frac{3}{2}$

5 Geometry

Source: 2007 AMC 10A #15

Four circles of radius 1 are each tangent to two sides of a square and externally tangent to a circle of radius 2, as shown. What is the area of the square?



- (A) 32 (B) $22 + 12\sqrt{2}$ (C) $16 + 16\sqrt{3}$ (D) 48 (E) $36 + 16\sqrt{2}$

Key idea. Tangent circles mean “join the centers”: the distance between centers is the sum of the radii. Here all the centers lie on the square’s diagonals, so the whole problem lives on one diagonal.

Step 1. Put the square with corners $(0, 0)$ and (s, s) . The small circle in the lower-left corner touches the two sides through $(0, 0)$, so its center is 1 from each: it is at $(1, 1)$. By symmetry the big circle is centered at the middle of the square, $(\frac{s}{2}, \frac{s}{2})$.

Step 2. The circles are externally tangent, so their centers are $2 + 1 = 3$ apart. Both centers lie on the diagonal $y = x$, and the distance between them is

$$\sqrt{2} \left(\frac{s}{2} - 1 \right) = 3, \quad \text{so} \quad \frac{s}{2} - 1 = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}, \quad s = 2 + 3\sqrt{2}.$$

Step 3. The area is

$$s^2 = (2 + 3\sqrt{2})^2 = 4 + 12\sqrt{2} + 18 = 22 + 12\sqrt{2}.$$

Another way. Walk along a diagonal from a corner to the center: corner to small center is the diagonal of a 1×1 square, $\sqrt{2}$; then 3 more to the big center. So the full diagonal is $2(3 + \sqrt{2}) = 6 + 2\sqrt{2}$, and a square’s area is half its diagonal squared:

$$\frac{(6 + 2\sqrt{2})^2}{2} = \frac{36 + 24\sqrt{2} + 8}{2} = 22 + 12\sqrt{2}.$$

Watch out. The corner of the square is *not* 1 away from the small center — that distance is $\sqrt{2}$. Using 1 gives a half-diagonal of $1 + 3 = 4$, a diagonal of 8, and area $\frac{8^2}{2} = 32$, choice (A).

Answer: (B) $22 + 12\sqrt{2}$

6 Counting & Probability

Source: 2003 AMC 10B #16

A restaurant offers three desserts, and exactly twice as many appetizers as main courses. A dinner consists of an appetizer, a main course, and a dessert. What is the least number of main courses that the restaurant should offer so that a customer could have a different dinner each night in the year 2003?

- (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Key idea. A dinner is three independent choices, so the number of different dinners is a product (the multiplication principle). “A different dinner each night” then says this product must be at least the number of nights in 2003.

Step 1. Let the restaurant offer m main courses. Then it offers $2m$ appetizers and 3 desserts, so the number of different dinners is

$$2m \cdot m \cdot 3 = 6m^2.$$

Step 2. The year 2003 is not a leap year (2003 is not divisible by 4), so it has 365 nights. We need

$$6m^2 \geq 365, \quad \text{that is,} \quad m^2 \geq \frac{365}{6} \approx 60.8.$$

Step 3. Test whole numbers: $7^2 = 49$ is too small, since $6 \cdot 49 = 294 < 365$. But $8^2 = 64$ works: $6 \cdot 64 = 384 \geq 365$. So the least number of main courses is 8.

For comparison, the choices give these numbers of dinners: $m = 4, 5, 6, 7, 8$ give 96, 150, 216, 294, 384. Only the last reaches 365.

Watch out. $\sqrt{60.8} \approx 7.8$, and dropping the decimal part gives 7 (choice D). But with 7 main courses there are only 294 dinners. In a “least number so that ...” question you must round *up*.

Answer: (E) 8

7 Algebra

Source: 2004 AMC 10A #17

Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

- (A) 250 (B) 300 (C) 350 (D) 400 (E) 500

Key idea. When two runners head towards each other, what matters is the distance they cover *together*. Between meetings that combined distance is a fixed fraction of the track, and with constant speeds each girl’s share is proportional to time — so compare the two stretches.

Let the track have length L .

Step 1. They start diametrically opposite, so along the track they are $\frac{L}{2}$ apart. Running towards each other, they meet when their distances add up to $\frac{L}{2}$. Brenda has run 100 meters of that.

Step 2. After meeting they run apart, and they meet again when together they have covered the whole track once more: a combined L . That is twice the combined distance of the first stretch, so (constant speeds) it takes twice as long, and Brenda runs $2 \cdot 100 = 200$ meters in this stretch.

Step 3. In the same stretch Sally runs 150 meters. Together they cover the whole track, so

$$L = 200 + 150 = 350.$$

Check: on a 350-meter track the first stretch has combined length 175, so Sally runs 75 while Brenda runs 100, a ratio of 4 : 3. In the second stretch Brenda runs 200 and Sally 150 — the same 4 : 3 ratio, as constant speeds require.

Watch out. If you overlook “diametrically opposite” and treat the first meeting like the second (combined distance L), Brenda runs 100 in each stretch and you get $L = 100 + 150 = 250$, choice (A).

Answer: (C) 350

8 Geometry

Source: 2002 AMC 10A #18

A $3 \times 3 \times 3$ cube is formed by gluing together 27 standard cubical dice. (On a standard die, the sum of the numbers on any pair of opposite faces is 7.) The smallest possible sum of all the numbers showing on the surface of the $3 \times 3 \times 3$ cube is

- (A) 60 (B) 72 (C) 84 (D) 90 (E) 96

Key idea. Each die can be turned independently, so minimize die by die. Sort the 27 dice by how many of their faces show on the surface; the rule about opposite faces tells you which numbers can show together.

Step 1. Sort the dice. In a $3 \times 3 \times 3$ cube there are 8 corner dice (3 faces showing), 12 edge dice, one in the middle of each edge (2 faces), 6 face-center dice (1 face), and 1 hidden die in the middle (0 faces). Check: $8 + 12 + 6 + 1 = 27$.

Step 2. Corner dice. The three showing faces are different faces of the die, so they carry three different numbers, and their sum is at least $1 + 2 + 3 = 6$. This is achievable: the opposite pairs are 1–6, 2–5, 3–4, so no two of 1, 2, 3 are opposite, and three faces of a cube with no opposite pair always meet at one corner. Total: $8 \cdot 6 = 48$.

Step 3. Edge dice. Two different faces show two different numbers, so the sum is at least $1 + 2 = 3$, and 1 and 2 are not opposite, so they are adjacent faces and can both show. Total: $12 \cdot 3 = 36$.

Step 4. Face-center dice show one face, at least 1: total 6. The middle die shows nothing.

Adding, the surface sum is at least $48 + 36 + 6 = 90$, and turning every die as described gives exactly 90.

Watch out. Forgetting the six face-center dice gives $48 + 36 = 84$, choice (C). Always check that your categories add up to 27 dice.

Answer: (D) 90

9 Number Theory

Source: 2018 AMC 10B #19

Joey and Chloe and their daughter Zoe all have the same birthday. Joey is 1 year older than Chloe, and Zoe is exactly 1 year old today. Today is the first of the 9 birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

- (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Key idea. Ages keep changing, but the *difference* between two ages never does. Writing Chloe's age as Zoe's age plus that fixed gap turns "a multiple of Zoe's age" into "Zoe's age divides the gap", so "9 birthdays" becomes "the gap has 9 divisors".

Step 1. Let g be the age gap between Chloe and Zoe; today Chloe is $g + 1$. On the birthday when Zoe turns z , Chloe is $z + g$, and

$$\frac{z + g}{z} = 1 + \frac{g}{z},$$

which is a whole number exactly when z divides g . Zoe's ages from today on are $z = 1, 2, 3, \dots$, so each divisor of g gives exactly one such birthday (today is $z = 1$). Hence g has exactly 9 divisors.

Step 2. If $g = p^a q^b \dots$, it has $(a + 1)(b + 1) \dots$ divisors. To get 9 we need $g = p^8$ or $g = p^2 q^2$. The smallest such numbers are $36 = 2^2 \cdot 3^2$, then $100 = 2^2 \cdot 5^2$, then larger ones. A mother of 101 with a one-year-old is impossible, so $g = 36$: Chloe is 37 and Joey is 38.

Check: Zoe's ages 1, 2, 3, 4, 6, 9, 12, 18, 36 (the divisors of 36) are the 9 birthdays.

Step 3. Joey's gap over Zoe is 37, so by the same argument Joey's age is a multiple of Zoe's exactly when z divides 37. Since 37 is prime, that happens only at $z = 1$ (today) and $z = 37$. The next time is when Zoe is 37 and Joey is $37 + 37 = 74$.

The sum of the digits is $7 + 4 = 11$.

Answer: (E) 11

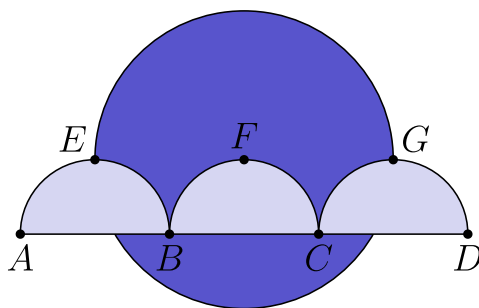
10 Geometry

Source: 2019 AMC 10B #20

As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, $\widehat{AEB}$, $\widehat{BFC}$, and $\widehat{CGD}$, have their diameters on $\overline{AD}$, lie in the same halfplane determined by line AD , and are tangent to line EG at E , F , and G , respectively. A circle of radius 2 has its center at F . The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

$$\frac{a}{b} \cdot \pi - \sqrt{c} + d,$$

where a , b , c , and d are positive integers and a and b are relatively prime. What is $a + b + c + d$?



- (A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Key idea. Cut the shaded region with the two horizontal lines EG and AD into pieces with familiar shapes. Line EG is the natural first cut because it passes through the center F , so it splits the big circle exactly in half.

The semicircles have radius 1 and touch line EG at their tops, so EG is parallel to AD and 1 unit above it; E , F , G sit directly above the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, with $EF = FG = 2$.

Step 1. Above EG . The semicircles reach line EG only at the points E , F , G , so the whole upper half of the big circle is shaded: $\frac{1}{2}\pi \cdot 2^2 = 2\pi$.

Step 2. Between EG and AD . Look at the 4×1 rectangle with top side $\overline{EG}$ and bottom side on $\overline{AD}$; its area is 4. Inside it the semicircles cover a quarter of $\widehat{AEB}$, all of $\widehat{BFC}$ and a quarter of $\widehat{CGD}$: $\frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4} = \pi$. The big circle does not reach outside this rectangle in this strip, and the small bottom corners of the rectangle that lie outside the big circle are covered by the outer semicircles. So the shaded area here is $4 - \pi$.

Step 3. Below AD . Let the big circle meet AD at P and Q , and let M be the midpoint of $\overline{BC}$, directly below F . Then $FM = 1$ and $FP = 2$, so $\cos \angle PFM = \frac{1}{2}$, $\angle PFM = 60^\circ$ and $\angle PFQ = 120^\circ$; also $PM = \sqrt{4 - 1} = \sqrt{3}$. The semicircles all lie above AD , so the whole circular segment below AD is shaded. It is a 120° sector minus triangle PFQ :

$$\frac{120}{360} \cdot \pi \cdot 2^2 - \frac{1}{2} \cdot 2\sqrt{3} \cdot 1 = \frac{4\pi}{3} - \sqrt{3}.$$

Step 4. Add. $2\pi + (4 - \pi) + \left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{7\pi}{3} - \sqrt{3} + 4$, so $a = 7$, $b = 3$, $c = 3$, $d = 4$ and $a + b + c + d = 17$.

Watch out. The 60° from $\cos \angle PFM = \frac{1}{2}$ is only *half* the central angle. Using a 60° sector gives a segment of $\frac{2\pi}{3} - \sqrt{3}$, a total of $\frac{5}{3}\pi - \sqrt{3} + 4$, and the sum 15, choice (C).

Answer: (E) 17
