
AMC 10

SET 13

HALF-LENGTH PRACTICE

SOLUTIONS

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All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	A	2018 AMC 10A #10	Algebra
2	C	2018 AMC 10B #11	Number Theory
3	D	2007 AMC 10A #12	Counting & Probability
4	D	2008 AMC 10A #13	Algebra
5	C	2010 AMC 10A #14	Geometry
6	E	2003 AMC 10B #15	Algebra
7	B	2015 AMC 10B #17	Geometry
8	C	2012 AMC 10B #18	Counting & Probability
9	A	2006 AMC 10B #19	Geometry
10	B	2008 AMC 10B #20	Counting & Probability

1 Algebra

Source: 2018 AMC 10A #10

Suppose that real number x satisfies

$$\sqrt{49 - x^2} - \sqrt{25 - x^2} = 3.$$

What is the value of

$$\sqrt{49 - x^2} + \sqrt{25 - x^2}?$$

- (A) 8 (B) $\sqrt{33} + 8$ (C) 9 (D) $2\sqrt{10} + 4$ (E) 12

Key idea. The given expression and the wanted one are *conjugates*: $a - b$ and $a + b$. Their product $a^2 - b^2$ has no square roots, and the x^2 cancels.

Let $a = \sqrt{49 - x^2}$ and $b = \sqrt{25 - x^2}$. We know $a - b = 3$ and want $a + b$.

Multiply:

$$(a - b)(a + b) = a^2 - b^2 = (49 - x^2) - (25 - x^2) = 24.$$

So $3 \cdot (a + b) = 24$, which gives $a + b = 8$.

Check that such an x really exists: from $a - b = 3$ and $a + b = 8$ we get $a = \frac{11}{2}$ and $b = \frac{5}{2}$. Then $49 - x^2 = \frac{121}{4}$ gives $x^2 = \frac{75}{4}$, and indeed $25 - \frac{75}{4} = \frac{25}{4} = b^2$. Everything is consistent.

There is no need to find x at all — the product trick skips it.

Answer: (A) 8

2 Number Theory

Source: 2018 AMC 10B #11

Which of the following expressions is never a prime number when p is a prime number?

- (A) $p^2 + 16$ (B) $p^2 + 24$ (C) $p^2 + 26$ (D) $p^2 + 46$ (E) $p^2 + 96$

Key idea. To show a number is *never* prime, find a small prime that *always* divides it. Squares have very few remainders mod 3, so test each constant mod 3.

Step 1. Squares of primes mod 3. If $p \neq 3$, then p is not a multiple of 3, so $p \equiv 1$ or $2 \pmod{3}$, and in both cases $p^2 \equiv 1 \pmod{3}$ (because $1^2 = 1$ and $2^2 = 4 \equiv 1$). If $p = 3$, then $p^2 = 9 \equiv 0 \pmod{3}$.

Step 2. Look at $p^2 + 26$. Since $26 \equiv 2 \pmod{3}$, for every prime $p \neq 3$ we get $p^2 + 26 \equiv 1 + 2 \equiv 0 \pmod{3}$. The number is a multiple of 3 and at least $4 + 26 = 30$, so it is not prime. For $p = 3$: $9 + 26 = 35 = 5 \cdot 7$, also not prime. So $p^2 + 26$ is never prime.

Step 3. The other four can be prime, so they are out:

$$5^2 + 16 = 41, \quad 7^2 + 24 = 73, \quad 5^2 + 46 = 71, \quad 19^2 + 96 = 457,$$

and all four results are prime. (Notice $16, 46 \equiv 1$ and $24, 96 \equiv 0 \pmod{3}$, so mod 3 does not rule them out.)

Watch out. Testing only $p = 2$ and $p = 3$ proves nothing: for $p = 2$ the five values are 20, 28, 30, 50, 100 and for $p = 3$ they are 25, 33, 35, 55, 105 — all composite. You need a reason that covers *every* prime.

Answer: (C) $p^2 + 26$

3 Counting & Probability

Source: 2007 AMC 10A #12

Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

- (A) 56 (B) 58 (C) 60 (D) 62 (E) 64

Key idea. Each tourist makes an independent two-way choice, so count all 2^6 assignments first, then throw away the few that break the “at least one tourist” rule. Counting everything and subtracting the bad cases is easier than counting the good cases directly.

Step 1. Ignore the stipulation. Each of the six tourists picks guide 1 or guide 2, so there are

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$$

possible assignments.

Step 2. Remove the bad ones. An assignment is bad only when one guide gets nobody, which means all six tourists chose the other guide. That happens in exactly 2 ways: everyone with guide 1, or everyone with guide 2.

Step 3. The number of valid groupings is $64 - 2 = 62$.

Another way. Count by the size of guide 1’s group, which can be 1, 2, 3, 4, or 5 tourists:

$$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} = 6 + 15 + 20 + 15 + 6 = 62.$$

Watch out. Forgetting the stipulation gives $2^6 = 64$, choice (E). The guides are different people, so a group of two with guide 1 is a different grouping from the same two with guide 2 — do not divide by 2.

Answer: (D) 62

4 Algebra

Source: 2008 AMC 10A #13

Doug can paint a room in 5 hours. Dave can paint the same room in 7 hours. Doug and Dave paint the room together and take a one-hour break for lunch. Let t be the total time, in hours, required for them to complete the job working together, including lunch. Which of the following equations is satisfied by t ?

- (A) $\left(\frac{1}{5} + \frac{1}{7}\right)(t+1) = 1$ (B) $\left(\frac{1}{5} + \frac{1}{7}\right)t + 1 = 1$ (C) $\left(\frac{1}{5} + \frac{1}{7}\right)t = 1$
 (D) $\left(\frac{1}{5} + \frac{1}{7}\right)(t-1) = 1$ (E) $(5+7)t = 1$

Key idea. Work problems run on *rates*: (rate) $\times$ (time actually spent working) = (fraction of the job done). The key is that t includes lunch, so the working time is not t .

Step 1. Rates. Doug paints the whole room in 5 hours, so he paints $\frac{1}{5}$ of the room per hour. Dave paints $\frac{1}{7}$ of the room per hour. Working together, their rates add: they paint $\frac{1}{5} + \frac{1}{7}$ of the room per hour.

Step 2. Working time. The total time t includes a one-hour lunch break during which nobody paints. So they actually paint for $t - 1$ hours.

Step 3. The whole room is one job, so

$$\left(\frac{1}{5} + \frac{1}{7}\right)(t-1) = 1.$$

As a check, $\frac{1}{5} + \frac{1}{7} = \frac{12}{35}$, so $t - 1 = \frac{35}{12}$ and $t = \frac{47}{12}$ hours, just under 4 hours — sensible, since together they are faster than Doug alone but lose an hour to lunch.

Watch out. Thinking “lunch adds an hour, so add 1” leads to $(t+1)$, choice (A). But t *already* contains the lunch hour; the painting time is the shorter quantity $t - 1$. Choice (C) forgets the break entirely.

Answer: (D) $\left(\frac{1}{5} + \frac{1}{7}\right)(t-1) = 1$

5 Geometry

Source: 2010 AMC 10A #14

Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD , and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

- (A) 60° (B) 75° (C) 90° (D) 105° (E) 120°

Key idea. Two equal angles plus a known 60° angle invite an angle chase: call the equal angles x and watch x cancel. The side condition $AB = 2 \cdot AC$ is only useful once an angle at A is known.

Step 1. Let $\angle BAE = \angle ACD = x$. Because $\triangle CFE$ is equilateral, $\angle CFE = 60^\circ$.

Step 2. Points A, F, E lie on one line, so $\angle AFC$ and $\angle CFE$ together make a straight angle:

$$\angle AFC = 180^\circ - 60^\circ = 120^\circ.$$

Step 3. In $\triangle AFC$, the angle at C is $\angle ACD = x$, so

$$\angle FAC = 180^\circ - 120^\circ - x = 60^\circ - x.$$

Since F is on $\overline{AE}$, this is $\angle EAC$. Therefore

$$\angle BAC = \angle BAE + \angle EAC = x + (60^\circ - x) = 60^\circ.$$

Step 4. Now use $AB = 2 \cdot AC$. Let M be the midpoint of $\overline{AB}$. Then $AM = AC$ and $\angle MAC = 60^\circ$, so $\triangle AMC$ is equilateral and $MC = MA = MB$. Thus C lies on the circle with diameter $\overline{AB}$, and $\angle ACB = 90^\circ$.

(Law of cosines check with $AC = 1$, $AB = 2$: $BC^2 = 4 + 1 - 2 \cdot 2 \cdot 1 \cdot \frac{1}{2} = 3$, and $1 + 3 = 4 = AB^2$.) The configuration really exists: with $x = 30^\circ$ everything fits.

Watch out. The angle chase ends with a 60° angle, and it is tempting to bubble 60° , choice (A). But that is $\angle BAC$, not the asked $\angle ACB$.

Answer: (C) 90°

6 Algebra

Source: 2003 AMC 10B #15

There are 100 players in a singles tennis tournament. The tournament is single elimination, meaning that a player who loses a match is eliminated. In the first round, the strongest 28 players are given a bye, and the remaining 72 players are paired off to play. After each round, the remaining players play in the next round. The match continues until only one player remains unbeaten. The total number of matches played is

- (A) a prime number (B) divisible by 2 (C) divisible by 5
(D) divisible by 7 (E) divisible by 11

Key idea. Instead of following the rounds, count *losers*. Every match produces exactly one loser, and every player except the champion loses exactly once. The details about byes are there to tempt you into round-by-round bookkeeping.

Step 1. Each match eliminates exactly one player, and each eliminated player is eliminated in exactly one match. So

$$\text{matches} = \text{eliminated players}.$$

Step 2. All 100 players except the one champion are eliminated, so $100 - 1 = 99$ matches are played.

Step 3. Test the choices: $99 = 9 \cdot 11 = 3^2 \cdot 11$. It is not prime, it is odd, and it is not a multiple of 5 or 7. It *is* divisible by 11.

Another way. Follow the rounds to confirm. Round 1: the 72 players give 36 matches, leaving 36 winners + 28 byes = 64 players. Then $64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$ uses $32 + 16 + 8 + 4 + 2 + 1 = 63$ matches. Total: $36 + 63 = 99$.

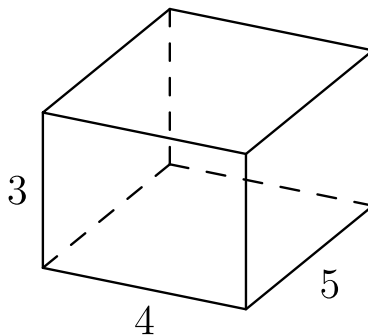
Watch out. If you lose track and count one match per player, 100 is divisible by both 2 and 5 — two answers at once, which is a sign the count is off.

Answer: (E) divisible by 11

7 Geometry

Source: 2015 AMC 10B #17

The centers of the faces of the right rectangular prism shown below are joined to create an octahedron. What is the volume of this octahedron?



- (A) $\frac{75}{12}$ (B) 10 (C) 12 (D) $10\sqrt{2}$ (E) 15

Key idea. An octahedron is two pyramids glued along a common base. Here the base is easy to spot: the centers of the four vertical faces all sit at half the prism's height, so they lie in one horizontal plane.

Step 1. The common base. The centers of the four vertical faces lie in the plane halfway up the prism. Their quadrilateral is a rhombus whose diagonals join centers of *opposite* faces, so the diagonals have lengths 4 and 5 (the length and width of the prism). A rhombus has area half the product of its diagonals:

$$\text{base area} = \frac{1}{2} \cdot 4 \cdot 5 = 10.$$

Step 2. The two apexes. The remaining vertices are the centers of the top and bottom faces. Each is $\frac{3}{2}$ above or below the middle plane, since the height is 3.

Step 3. Volume. Each pyramid has volume $\frac{1}{3} \cdot (\text{base}) \cdot (\text{height})$, so

$$V = 2 \cdot \frac{1}{3} \cdot 10 \cdot \frac{3}{2} = 10.$$

Another way. This works for any box: the rhombus is half of the 4×5 cross-section and the two pyramids have total height 3, so $V = \frac{1}{3} \cdot \frac{4 \cdot 5}{2} \cdot 3 = \frac{4 \cdot 5 \cdot 3}{6}$. The octahedron is always $\frac{1}{6}$ of the box, here $\frac{60}{6} = 10$.

Watch out. The base is a rhombus, not the full 4×5 rectangle. Using area $4 \cdot 5 = 20$ doubles the answer to 20; using the full height 3 for each pyramid doubles it too.

Answer: (B) 10

8 Counting & Probability

Source: 2012 AMC 10B #18

Suppose that one of every 500 people in a certain population has a particular disease, which displays no symptoms. A blood test is available for screening for this disease. For a person who has this disease, the test always turns out positive.

For a person who does not have the disease, however, there is a 2% false positive rate. In other words, for such people, 98% of the time the test will turn out negative, but 2% of the time the test will turn out positive and will incorrectly indicate that the person has the disease.

Let p be the probability that a person who is chosen at random from this population and gets a positive test result actually has the disease. Which of the following is closest to p ?

- (A) $\frac{1}{98}$ (B) $\frac{1}{9}$ (C) $\frac{1}{11}$ (D) $\frac{49}{99}$ (E) $\frac{98}{99}$

Key idea. p is a probability *among people who tested positive*. So picture an actual crowd, count everyone who tests positive, and ask what fraction of them is sick. Concrete numbers beat formulas here.

Step 1. Picture 500 people. On average 1 of them has the disease, and that person tests positive.

Step 2. The other 499 are healthy, and 2% of them test positive anyway:

$$0.02 \cdot 499 = 9.98 \approx 10 \text{ false positives.}$$

Step 3. So about $1 + 10 = 11$ people test positive, and only 1 of them is sick. Hence $p \approx \frac{1}{11}$.

Exactly:

$$p = \frac{1}{1 + 9.98} = \frac{1}{10.98} = \frac{50}{549} \approx 0.0911,$$

while $\frac{1}{11} \approx 0.0909$ and $\frac{1}{9} \approx 0.111$. The closest choice is $\frac{1}{11}$.

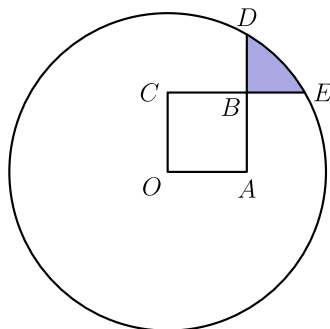
Watch out. “The test is 98% accurate, so a positive result means you are 98% likely to be sick” points to $\frac{98}{99}$, choice (E). It ignores how rare the disease is: healthy people outnumber sick ones 499 to 1, so even a small false-positive rate produces ten times more false alarms than true cases.

Answer: (C) $\frac{1}{11}$

9 Geometry

Source: 2006 AMC 10B #19

A circle of radius 2 is centered at O . Square $OABC$ has side length 1. Sides $\overline{AB}$ and $\overline{CB}$ are extended past B to meet the circle at D and E , respectively. What is the area of the shaded region in the figure, which is bounded by $\overline{BD}$, $\overline{BE}$, and the minor arc connecting D and E ?



- (A) $\frac{\pi}{3} + 1 - \sqrt{3}$ (B) $\frac{\pi}{2}(2 - \sqrt{3})$ (C) $\pi(2 - \sqrt{3})$ (D) $\frac{\pi}{6} + \frac{\sqrt{3} - 1}{2}$ (E) $\frac{\pi}{3} - 1 + \sqrt{3}$

Key idea. A region with a curved side is almost always “a sector minus straight-sided pieces.” Draw the radii $\overline{OD}$ and $\overline{OE}$ and the segment $\overline{OB}$: the sector DOE splits into the shaded region and two triangles. A radius of 2 against a distance of 1 signals 30–60–90 triangles.

Step 1. Locate D and E . Put $O = (0, 0)$, $A = (1, 0)$, $B = (1, 1)$, $C = (0, 1)$. Then D lies on the line $x = 1$ with $OD = 2$, so $D = (1, \sqrt{3})$; similarly $E = (\sqrt{3}, 1)$. In right triangle OAD , $OA = 1$ and $OD = 2$, so $\angle AOD = 60^\circ$. Likewise $\angle COE = 60^\circ$, so $\angle AOE = 30^\circ$ and

$$\angle DOE = 60^\circ - 30^\circ = 30^\circ.$$

Step 2. The sector. Its area is $\frac{30}{360} \cdot \pi \cdot 2^2 = \frac{\pi}{3}$.

Step 3. The triangles. Segments $\overline{BD}$ and $\overline{BE}$ cut the sector into the shaded region and quadrilateral $ODBE$, and the segment $\overline{OB}$ splits that quadrilateral into triangles OBD and OBE . In $\triangle OBD$, the base $BD = \sqrt{3} - 1$ lies on the line $x = 1$, which is at distance 1 from O , so its area is $\frac{1}{2}(\sqrt{3} - 1) \cdot 1$. By symmetry $\triangle OBE$ has the same area. Together: $\sqrt{3} - 1$.

Step 4. Shaded area:

$$\frac{\pi}{3} - (\sqrt{3} - 1) = \frac{\pi}{3} + 1 - \sqrt{3} \approx 0.315.$$

Watch out. Adding the triangles instead of subtracting them gives $\frac{\pi}{3} - 1 + \sqrt{3}$, choice (E). That value is about 1.78, far more than the small sliver in the picture.

Answer: (A) $\frac{\pi}{3} + 1 - \sqrt{3}$

10 Counting & Probability

Source: 2008 AMC 10B #20

The faces of a cubical die are marked with the numbers 1, 2, 2, 3, 3, and 4. The faces of a second cubical die are marked with the numbers 1, 3, 4, 5, 6, and 8. Both dice are thrown. What is the probability that the sum of the two top numbers will be 5, 7, or 9?

- (A) $\frac{5}{18}$ (B) $\frac{7}{18}$ (C) $\frac{11}{18}$ (D) $\frac{3}{4}$ (E) $\frac{8}{9}$

Key idea. There are $6 \cdot 6 = 36$ equally likely outcomes, one for each pair of *faces*. The repeated numbers 2, 2 and 3, 3 are different faces, so each pair using them counts twice. Then just list the pairs for each target sum.

Step 1. Sum 5. Pairs (first die, second die): (1, 4), (2, 3), (4, 1). The pair (2, 3) occurs twice (two faces show 2), so there are $1 + 2 + 1 = 4$ outcomes.

Step 2. Sum 7: (1, 6), (2, 5), (3, 4), (4, 3). Both (2, 5) and (3, 4) count twice, giving $1 + 2 + 2 + 1 = 6$ outcomes.

Step 3. Sum 9: (1, 8), (3, 6), (4, 5). The pair (3, 6) counts twice, giving $1 + 2 + 1 = 4$ outcomes.

Step 4. In total $4 + 6 + 4 = 14$ favorable outcomes, so the probability is $\frac{14}{36} = \frac{7}{18}$.

Another way. Use parity. Each die has three odd faces and three even faces, so the sum is odd with probability $\frac{1}{2}$, i.e. in 18 outcomes. The odd sums are 3, 5, 7, 9, 11. Sum 3 needs (2, 1): 2 outcomes. Sum 11 needs (3, 8): 2 outcomes. So sums 5, 7, 9 occur in $18 - 2 - 2 = 14$ outcomes, and the probability is $\frac{14}{36} = \frac{7}{18}$.

Watch out. Listing each pair of *numbers* only once gives $3 + 4 + 3 = 10$ outcomes and $\frac{10}{36} = \frac{5}{18}$, choice (A). The two 2s and two 3s are separate faces.

Answer: (B) $\frac{7}{18}$