
AMC 10

SET 14

HALF-LENGTH PRACTICE

SOLUTIONS

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All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	E	2006 AMC 10A #10	Algebra
2	A	2009 AMC 10B #11	Counting & Probability
3	C	2013 AMC 10A #12	Geometry
4	A	2019 AMC 10B #13	Algebra
5	D	2019 AMC 10A #14	Counting & Probability
6	C	2001 AMC 10 #15	Geometry
7	B	2002 AMC 10A #16	Algebra
8	D	2014 AMC 10B #17	Number Theory
9	C	2002 AMC 10B #19	Algebra
10	A	2012 AMC 10B #20	Counting & Probability

1 Algebra

Source: 2006 AMC 10A #10

For how many real values of x is $\sqrt{120 - \sqrt{x}}$ an integer?

- (A) 3 (B) 6 (C) 9 (D) 10 (E) 11

Key idea. Do not hunt for x directly. Name the integer the expression must equal, find which integers are possible, and then work backwards to x .

Suppose $\sqrt{120 - \sqrt{x}} = k$, where k is an integer. A square root is never negative, so $k \geq 0$. Squaring gives $120 - \sqrt{x} = k^2$, so $\sqrt{x} = 120 - k^2$.

Since $\sqrt{x} \geq 0$, we need $k^2 \leq 120$. As $10^2 = 100 \leq 120 < 121 = 11^2$, the possible values are $k = 0, 1, 2, \dots, 10$, which is 11 values.

Each k gives exactly one $x = (120 - k^2)^2$. For example, $k = 10$ gives $x = 20^2 = 400$ and $k = 0$ gives $x = 120^2 = 14400$. The numbers $120 - k^2$ are different nonnegative numbers for different k , so their squares differ and these 11 values of x are all different.

Watch out. Do not forget $k = 0$, which gives $x = 14400$. Counting only $k = 1, \dots, 10$ gives 10, choice (D).

Answer: (E) 11**2** Counting & Probability

Source: 2009 AMC 10B #11

How many 7-digit palindromes (numbers that read the same backward as forward) can be formed using the digits 2, 2, 3, 3, 5, 5, 5?

- (A) 6 (B) 12 (C) 24 (D) 36 (E) 48

Key idea. In a palindrome the digits come in mirror pairs, except the middle one. So first ask which digit can sit in the middle — the digit counts 2, 2, 3, 3, 5, 5, 5 decide that for you.

Step 1. Number the places 1 to 7. Reading backward must give the same number, so place 1 equals place 7, place 2 equals place 6, and place 3 equals place 5. Only place 4, the middle, has no partner.

Step 2. The six outer places use digits in matching pairs, so every digit is used an even number of times there. The digit 2 appears twice and 3 appears twice, but 5 appears three times. The extra 5 must go in the middle. So place 4 is 5.

Step 3. What remains is 2, 2, 3, 3, 5, 5: one pair of each digit. The three pairs go to the three mirror positions (1, 7), (2, 6), (3, 5), one pair per position. That is choosing an order of 2, 3, 5 for places 1, 2, 3, and there are $3! = 3 \cdot 2 \cdot 1 = 6$ orders.

For example, 2355532, 2535352 and 5325235 are three of them.

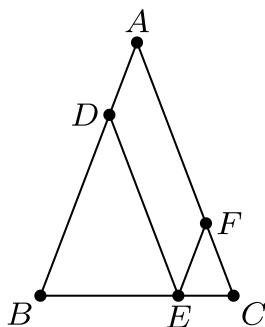
Watch out. A palindrome is fixed by its first four digits, so it is tempting to count the orders of 2, 3, 5, 5 in places 1–4: $\frac{4!}{2!} = 12$, choice (B). But that lets a 2 or 3 sit in the middle; for example 3 in place 4 forces the number $abc3cba$ with $\{a, b, c\} = \{2, 5, 5\}$, which uses 5 four times and 3 only once — not the given digits. Only the orders with 5 in place 4 work.

Answer: (A) 6

3 Geometry

Source: 2013 AMC 10A #12

In $\triangle ABC$, $AB = AC = 28$ and $BC = 20$. Points D , E , and F are on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of parallelogram $ADEF$?



- (A) 48 (B) 52 (C) 56 (D) 60 (E) 72

Key idea. The problem never says where E is, so the perimeter cannot depend on it. Parallel lines in an isosceles triangle create smaller isosceles triangles; use them to trade the sides DE and EF for pieces of $\overline{AB}$ and $\overline{AC}$.

Step 1. Since $\overline{DE} \parallel \overline{AC}$, the angles of $\triangle DBE$ match those of $\triangle ABC$: $\angle DBE = \angle ABC$ and $\angle DEB = \angle ACB$. Because $AB = AC$, we have $\angle ABC = \angle ACB$, so $\angle DBE = \angle DEB$. Equal base angles mean $\triangle DBE$ is isosceles with $DE = DB$.

Step 2. In the same way, $\overline{EF} \parallel \overline{AB}$ gives $\angle FEC = \angle ABC = \angle ACB = \angle FCE$, so $\triangle FEC$ is isosceles with $EF = FC$.

Step 3. Now add up the four sides of $ADEF$, replacing DE by DB and EF by FC :

$$AD + DE + EF + FA = (AD + DB) + (FC + FA) = AB + AC.$$

So the perimeter is $28 + 28 = 56$, wherever E is placed on $\overline{BC}$.

Notice that $BC = 20$ was never used — it is there only to fix the shape.

Another way. Since the answer cannot depend on where E is, choose the easiest spot: the midpoint of $\overline{BC}$. Then D and F are midpoints of $\overline{AB}$ and $\overline{AC}$, every side of $ADEF$ is half of 28, and the perimeter

is $4 \cdot 14 = 56$.

Answer: (C) 56

4 Algebra

Source: 2019 AMC 10B #13

What is the sum of all real numbers x for which the median of the numbers 4, 6, 8, 17, and x is equal to the mean of those five numbers?

- (A) -5 (B) 0 (C) 5 (D) $\frac{15}{4}$ (E) $\frac{35}{4}$

Key idea. The mean is a simple formula in x , but the median depends on where x falls among 4, 6, 8, 17. So split into cases by the position of x , solve in each case, and keep only the solutions that really lie in their case.

Step 1. The mean is

$$\frac{4 + 6 + 8 + 17 + x}{5} = \frac{35 + x}{5}.$$

Step 2. The median is the middle (third) number once the five are sorted.

- If $x \leq 6$, the sorted list has 6 in the middle, so the median is 6.
- If $6 \leq x \leq 8$, then x itself is in the middle, so the median is x .
- If $x \geq 8$, the median is 8.

Step 3. Solve each case.

Case $x \leq 6$: $\frac{35+x}{5} = 6$ gives $35 + x = 30$, so $x = -5$. This satisfies $x \leq 6$, so it works. Check: the list $-5, 4, 6, 8, 17$ has median 6 and mean $\frac{30}{5} = 6$.

Case $6 \leq x \leq 8$: $\frac{35+x}{5} = x$ gives $35 + x = 5x$, so $x = \frac{35}{4} = 8.75$. That is bigger than 8, so it is not in this case — reject it.

Case $x \geq 8$: $\frac{35+x}{5} = 8$ gives $x = 5$, which is less than 8 — reject it.

So $x = -5$ is the only solution, and the sum is -5 .

Watch out. Forgetting to check each solution against its case. Adding all three candidates gives $-5 + \frac{35}{4} + 5 = \frac{35}{4}$, choice (E).

Answer: (A) -5

5 Counting & Probability

Source: 2019 AMC 10A #14

For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N ?

- (A) 14 (B) 16 (C) 18 (D) 19 (E) 21

Key idea. Whether two lines meet depends only on whether they are parallel. So sort the four lines into groups of parallel lines; there are only five ways to split 4, and each one is easy to count. This gives every possible N at once — no guessing.

Lines in the same group never meet; lines in different groups meet exactly once.

Groups 4. All four parallel: no intersections, $N = 0$.

Groups 3 + 1. The lone line crosses the three parallel lines at three different points, and nothing else meets: $N = 3$.

Groups 2 + 2. Each line of one pair crosses both lines of the other pair, giving 4 points. They are all different (two of them on the same line come from two different parallel lines). $N = 4$.

Groups 2 + 1 + 1. Call the parallel pair p_1, p_2 and the others m, n . Then m and n each cross p_1 and p_2 , and m meets n : five crossings. If m and n meet on p_1 (or p_2), three crossings merge into one point and $N = 3$; otherwise all five are different and $N = 5$.

Groups 1 + 1 + 1 + 1. No two are parallel, so all $\binom{4}{2} = 6$ pairs meet. If no three lines pass through one point, $N = 6$. If exactly three pass through a point P , their 3 pairs share P , and the fourth line meets them at 3 other points: $N = 4$. If all four pass through one point, $N = 1$. (Two different triples share two lines, hence the same point, so “two triple points” forces all four through one point.)

So N can be 0, 1, 3, 4, 5, 6 and nothing else; in particular $N = 2$ never happens. The sum is $0 + 1 + 3 + 4 + 5 + 6 = 19$.

Watch out. Assuming every value from 0 to 6 occurs gives $0 + 1 + 2 + \cdots + 6 = 21$, choice (E). The value $N = 2$ appears in no case above.

Answer: (D) 19

6 Geometry

Source: 2001 AMC 10 #15

A street has parallel curbs 40 feet apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet and each stripe is 50 feet long. Find the distance, in feet, between the stripes.

- (A) 9 (B) 10 (C) 12 (D) 15 (E) 25

Key idea. The crosswalk is a parallelogram, and you are told one base with its height and the length of the other base. Whenever a parallelogram has two bases and you need a height, compute its area two ways.

Step 1. The crosswalk has two sides on the curbs (each 15 feet long) and two sides along the stripes (each 50 feet long). Opposite sides are parallel, so it is a parallelogram.

Step 2. Take a 15-foot curb side as the base. The height is the perpendicular distance between the curbs, which is 40 feet. So the area is

$$15 \cdot 40 = 600 \text{ square feet.}$$

Step 3. Now take a 50-foot stripe as the base. The matching height is the perpendicular distance d between the stripes — exactly what we want. So $50d = 600$, and $d = \frac{600}{50} = 12$.

It is reasonable: the stripes cross at a slant, so the gap measured straight across (12) is shorter than the gap measured along the curb (15).

Watch out. The 15 feet is measured *along the curb*, which is slanted relative to the stripes, so it is not the distance between them. Answering 15 is choice (D).

Answer: (C) 12

7 Algebra

Source: 2002 AMC 10A #16

If $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$, then $a + b + c + d$ is

- (A) -5 (B) $-\frac{10}{3}$ (C) $-\frac{7}{3}$ (D) $\frac{5}{3}$ (E) 5

Key idea. A long chain of equal expressions is best handled by giving the common value a single name, say k . Then every variable can be written in terms of k , and the last link of the chain becomes one equation in k .

Step 1. Let k be the common value. Then

$$a = k - 1, \quad b = k - 2, \quad c = k - 3, \quad d = k - 4.$$

Step 2. Add them:

$$a + b + c + d = 4k - (1 + 2 + 3 + 4) = 4k - 10.$$

Step 3. The last expression in the chain also equals k :

$$a + b + c + d + 5 = k.$$

Substituting, $4k - 10 + 5 = k$, so $4k - 5 = k$, which gives $3k = 5$ and $k = \frac{5}{3}$.

Step 4. The question asks for $a + b + c + d$, not k . From Step 3, $a + b + c + d = k - 5$, so

$$a + b + c + d = \frac{5}{3} - 5 = \frac{5}{3} - \frac{15}{3} = -\frac{10}{3}.$$

Check: $a = \frac{2}{3}$, $b = -\frac{1}{3}$, $c = -\frac{4}{3}$, $d = -\frac{7}{3}$. Each of $a + 1$, $b + 2$, $c + 3$, $d + 4$ equals $\frac{5}{3}$, and $a + b + c + d + 5 = -\frac{10}{3} + 5 = \frac{5}{3}$. ✓

Another way. Work with $S = a + b + c + d$ directly. Adding the four equations $a + 1 = k$, $b + 2 = k$, $c + 3 = k$, $d + 4 = k$ gives $S + 10 = 4k$, and the last link says $k = S + 5$. So

$$S + 10 = 4(S + 5) = 4S + 20,$$

hence $-3S = 10$ and $S = -\frac{10}{3}$, with no need to find k at all.

Watch out. Stopping once you find $k = \frac{5}{3}$. That is the common value, not $a + b + c + d$; it is choice (D).

Answer: (B) $-\frac{10}{3}$

8 Number Theory

Source: 2014 AMC 10B #17

What is the greatest power of 2 that is a factor of $10^{1002} - 4^{501}$?

- (A) 2^{1002} (B) 2^{1003} (C) 2^{1004} (D) 2^{1005} (E) 2^{1006}

Key idea. Both terms are full of factors of 2, so pull out the common power first. What is left is $5^{1002} - 1$, a difference of squares — and factoring it is the way to see exactly how many more 2's are hiding.

Step 1. Write both terms with base 2 visible: $10^{1002} = 2^{1002} \cdot 5^{1002}$ and $4^{501} = (2^2)^{501} = 2^{1002}$. So

$$10^{1002} - 4^{501} = 2^{1002} (5^{1002} - 1).$$

Step 2. Since $1002 = 2 \cdot 501$, use a difference of squares:

$$5^{1002} - 1 = (5^{501} - 1)(5^{501} + 1).$$

Step 3. Count the 2's in each factor by looking at remainders. Note $5^2 = 25$ leaves remainder 1 when divided by 8. Since 501 is odd, $5^{501} = 5 \cdot (5^2)^{250}$ leaves remainder 5 when divided by 8.

- $5^{501} - 1$ leaves remainder 4 when divided by 8: divisible by 4 but not 8. That is exactly 2^2 .
- $5^{501} + 1$ leaves remainder 6 when divided by 8, hence remainder 2 when divided by 4: divisible by 2 but not 4. That is exactly 2^1 .

Step 4. So $5^{1002} - 1$ contains exactly $2^{2+1} = 2^3$, and the whole number contains exactly

$$2^{1002} \cdot 2^3 = 2^{1005}.$$

Watch out. Noticing only that $5^{1002} - 1$ is divisible by $5 - 1 = 4$ and stopping there gives $2^{1002} \cdot 2^2 = 2^{1004}$, choice (C). Factoring shows a third 2 hiding in $5^{501} + 1$.

Answer: (D) 2^{1005}

9 Algebra

Source: 2002 AMC 10B #19

Suppose that $\{a_n\}$ is an arithmetic sequence with

$$a_1 + a_2 + \cdots + a_{100} = 100$$

and

$$a_{101} + a_{102} + \cdots + a_{200} = 200.$$

What is the value of $a_2 - a_1$?

- (A) 0.0001 (B) 0.001 (C) 0.01 (D) 0.1 (E) 1

Key idea. The two sums cover blocks of the same length, 100 terms each, exactly 100 places apart. So pair each term of the second block with the term 100 places earlier; each pair differs by the same amount, and the difference of the sums falls out.

Let $d = a_2 - a_1$ be the common difference.

Step 1. Moving 100 places along an arithmetic sequence adds d a hundred times, so

$$a_{k+100} = a_k + 100d \quad \text{for every } k.$$

For example, $a_{101} = a_1 + 100d$ and $a_{200} = a_{100} + 100d$.

Step 2. Write the second sum term by term:

$$a_{101} + \cdots + a_{200} = (a_1 + 100d) + \cdots + (a_{100} + 100d).$$

There are 100 terms, each $100d$ bigger than its partner, so

$$a_{101} + \cdots + a_{200} = (a_1 + \cdots + a_{100}) + 100 \cdot 100d.$$

Step 3. Substituting the given sums: $200 = 100 + 10000d$, so $10000d = 100$ and $d = \frac{100}{10000} = 0.01$.

(It is consistent: $a_1 = 0.505$ makes the first sum 100.)

Another way. Use averages. In an arithmetic sequence the average of a block of consecutive terms is the average of its first and last terms. The first block has average $\frac{100}{100} = 1$, so $\frac{a_1 + a_{100}}{2} = 1$; the second has average $\frac{200}{100} = 2$, so $\frac{a_{101} + a_{200}}{2} = 2$. Each of a_{101} , a_{200} is $100d$ more than a_1 , a_{100} , so the second average is $100d$ more than the first: $100d = 2 - 1 = 1$ and $d = 0.01$.

Watch out. Writing the second sum as the first sum plus $100d$ — adding the shift once instead of once per term — gives $200 = 100 + 100d$, so $d = 1$, choice (E).

Answer: (C) 0.01

10 Counting & Probability

Source: 2012 AMC 10B #20

Bernardo and Silvia play the following game. An integer between 0 and 999 inclusive is selected and given to Bernardo. Whenever Bernardo receives a number, he doubles it and passes the result to Silvia. Whenever Silvia receives a number, she adds 50 to it and passes the result to Bernardo. The winner is the last person who produces a number less than 1000.

Let N be the smallest initial number that results in a win for Bernardo. What is the sum of the digits of N ?

- (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Key idea. The moves are completely fixed — no choices are made — so just write the numbers in terms of the starting value x . Bernardo wins exactly when one of his numbers is below 1000 but Silvia's next one is not.

Step 1. Start with x . The numbers produced are

$$\begin{array}{ll} \text{B: } 2x, & \text{S: } 2x + 50, \\ \text{B: } 4x + 100, & \text{S: } 4x + 150, \\ \text{B: } 8x + 300, & \text{S: } 8x + 350, \\ \text{B: } 16x + 700, & \text{S: } 16x + 750, \\ \text{B: } 32x + 1500. & \end{array}$$

The numbers never decrease, so the game stops the first time a number reaches 1000. The last number listed is at least 1500, so the game always ends by then.

Step 2. Bernardo wins when one of his numbers is below 1000 and Silvia's next is 1000 or more. Since Silvia adds 50, Bernardo's number must be from 950 to 999. Check each of his rows:

- $950 \leq 2x \leq 999$: $x = 475, \dots, 499$.
- $950 \leq 4x + 100 \leq 999$: $x = 213, \dots, 224$.
- $950 \leq 8x + 300 \leq 999$: $x = 82, \dots, 87$.
- $950 \leq 16x + 700 \leq 999$: $250 \leq 16x \leq 299$, so $x = 16, 17, 18$.

Step 3. The smallest winning start is $N = 16$. Check: $16 \rightarrow 32 \rightarrow 82 \rightarrow 164 \rightarrow 214 \rightarrow 428 \rightarrow 478 \rightarrow 956 \rightarrow 1006$. Bernardo produced 956, the last number below 1000, so he wins. For $x \leq 15$, Silvia's $16x + 750 \leq 990$ is still below 1000 and then Bernardo's $32x + 1500$ is not, so Silvia wins.

The digit sum is $1 + 6 = 7$.

Watch out. Stopping after Bernardo's first turn: the window 475–499 is not the smallest. Its least value 475 has digit sum 16, which is not a choice — a sign to keep going.

Answer: (A) 7