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# AMC 10

# SET 15

HALF-LENGTH PRACTICE

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# SOLUTIONS

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[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

## Answer Key and Sources

| No. | Ans. | Source           | Topic                  |
|-----|------|------------------|------------------------|
| 1   | C    | 2015 AMC 10A #10 | Counting & Probability |
| 2   | D    | 2009 AMC 10A #11 | Geometry               |
| 3   | A    | 2008 AMC 10B #12 | Algebra                |
| 4   | D    | 2011 AMC 10B #13 | Counting & Probability |
| 5   | C    | 2000 AMC 10 #14  | Number Theory          |
| 6   | E    | 2009 AMC 10B #15 | Algebra                |
| 7   | B    | 2013 AMC 10B #16 | Geometry               |
| 8   | D    | 2007 AMC 10A #17 | Number Theory          |
| 9   | E    | 2012 AMC 10A #18 | Geometry               |
| 10  | C    | 2004 AMC 10B #19 | Algebra                |

**1** Counting & Probability

Source: 2015 AMC 10A #10

How many rearrangements of  $abcd$  are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either  $ab$  or  $ba$ .

- (A) 0                      (B) 1                      (C) 2                      (D) 3                      (E) 4

**Key idea.** Only 24 arrangements exist, so organise them by their first letter and see which ones survive.

The forbidden neighbours are the pairs  $a$ – $b$ ,  $b$ – $c$  and  $c$ – $d$  (in either order).

*Starting with  $a$ :* the next letter cannot be  $b$ , so it is  $c$  or  $d$ . After  $ac$ , the letter  $c$  may not be followed by  $b$  or  $d$  — but those are the only letters left. After  $ad$ , the last two letters  $b$  and  $c$  end up side by side. So nothing starts with  $a$ , and by the same reasoning (swap  $a \leftrightarrow d$ ,  $b \leftrightarrow c$ ) nothing starts with  $d$ .

*Starting with  $b$ :* the next letter must be  $d$  (not  $a$  or  $c$ ). After  $d$  we need  $a$  (not  $c$ ), and then  $c$  ends it:  $bdac$ . Check:  $bd$ ,  $da$ ,  $ac$  are all allowed.

*Starting with  $c$ :* by the same swap, the only option is  $cadb$ .

So there are exactly 2 rearrangements:  $bdac$  and  $cadb$ .

**Answer: (C) 2****2** Geometry

Source: 2009 AMC 10A #11

One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?

- (A) 8                      (B) 27                      (C) 64                      (D) 125                      (E) 216

**Key idea.** Name the side length and write both volumes. The phrase “increased by 1 ... decreased by 1” is a signal for the difference of squares  $(x + 1)(x - 1) = x^2 - 1$ .

**Step 1.** Let the cube have side length  $x$ , so its volume is  $x^3$ .

**Step 2.** The new solid measures  $(x + 1)$  by  $(x - 1)$  by  $x$ . Its volume is

$$x(x + 1)(x - 1) = x(x^2 - 1) = x^3 - x.$$

**Step 3.** So the new solid is exactly  $x$  smaller than the cube. We are told it is 5 smaller, so  $x^3 - x = x^3 - 5$ , which gives  $x = 5$ .

**Step 4.** The cube’s volume is  $5^3 = 125$ .

Check: the new solid is  $6 \times 4 \times 5 = 120$ , and  $125 - 120 = 5$ . ✓

Notice what Step 3 says in general: whatever the cube, this change always shrinks the volume by exactly the side length. That makes it easy to see why no other choice works — for instance a cube of volume 64 (side 4) would lose only 4.

**Watch out.** The equation gives the side length  $x = 5$ , but the question asks for the *volume*. Stopping at 5 leaves you looking for a choice that is not there; the answer is  $5^3$ .

**Answer: (D)** 125

### 3 Algebra

Source: 2008 AMC 10B #12

Postman Pete has a pedometer to count his steps. The pedometer records up to 99999 steps, then flips over to 00000 on the next step. Pete plans to determine his mileage for a year. On January 1 Pete sets the pedometer to 00000. During the year, the pedometer flips from 99999 to 00000 forty-four times. On December 31 the pedometer reads 50000. Pete takes 1800 steps per mile. Which of the following is closest to the number of miles Pete walked during the year?

- (A) 2500                      (B) 3000                      (C) 3500                      (D) 4000                      (E) 4500

**Key idea.** Think of the pedometer like a car's odometer: each flip means one complete lap of 100000 steps. Count the full laps, add the final reading, then convert to miles.

**Step 1. How many steps does one flip represent?** The display goes 00000, 00001, ..., 99999 and then the next step returns it to 00000. That is 100000 steps per full cycle (not 99999 — the step that causes the flip counts too).

**Step 2. Total steps.** Forty-four full cycles plus the 50000 showing at the end:

$$44 \times 100\,000 + 50\,000 = 4\,400\,000 + 50\,000 = 4\,450\,000 \text{ steps.}$$

**Step 3. Convert to miles.** Divide by 1800 steps per mile:

$$\frac{4\,450\,000}{1800} = \frac{44\,500}{18} \approx 2472.$$

**Step 4. Compare with the choices.** The value 2472 is just 28 away from 2500 and more than 500 from every other choice, so 2500 is the closest.

A quick check without a calculator:  $4\,450\,000 \div 2000 = 2225$ . Dividing by 1800 instead of 2000 multiplies the result by  $\frac{2000}{1800} = \frac{10}{9}$ , which adds one-ninth:  $2225 + 247 \approx 2472$ .

Because the question asks only for the closest choice, small slips (using 99999 per flip, or even forgetting the final 50000, which gives about 2444) would not change the answer here — but getting the number of zeros right in Step 2 matters a lot.

**Answer: (A)** 2500

## 4 Counting &amp; Probability

Source: 2011 AMC 10B #13

Two real numbers are selected independently at random from the interval  $[-20, 10]$ . What is the probability that the product of those numbers is greater than zero?

- (A)  $\frac{1}{9}$                       (B)  $\frac{1}{3}$                       (C)  $\frac{4}{9}$                       (D)  $\frac{5}{9}$                       (E)  $\frac{2}{3}$

**Key idea.** A product of two numbers is positive exactly when the two numbers have the *same sign*. So only the sign of each number matters, and the length of the interval tells you how likely each sign is.

**Step 1. Probability of each sign.** The interval  $[-20, 10]$  has length 30. The negative part, from  $-20$  to  $0$ , has length 20; the positive part, from  $0$  to  $10$ , has length 10. So for one number

$$P(\text{negative}) = \frac{20}{30} = \frac{2}{3}, \quad P(\text{positive}) = \frac{10}{30} = \frac{1}{3}.$$

Picking exactly 0 has probability 0, so we can ignore it.

**Step 2. Same sign.** The two numbers are chosen independently, so we multiply:

$$P(\text{both negative}) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}, \quad P(\text{both positive}) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

**Step 3. Add the two cases.** These cases cannot happen together, so

$$P(\text{product} > 0) = \frac{4}{9} + \frac{1}{9} = \frac{5}{9}.$$

**Another way.** Count the opposite event. The product is negative when one number is negative and the other positive; that happens in two orders, so its probability is  $2 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9}$ . Then  $1 - \frac{4}{9} = \frac{5}{9}$ .

**Watch out.** “Greater than zero” is easy to misread as “both numbers are positive.” That case alone gives  $\frac{1}{9}$ , choice (A). Two negatives also make a positive product — and here that is the *bigger* case,  $\frac{4}{9}$ , choice (C). You need both.

**Answer: (D)**  $\frac{5}{9}$

## 5 Number Theory

Source: 2000 AMC 10 #14

Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walter entered?

- (A) 71                      (B) 76                      (C) 80                      (D) 82                      (E) 91

**Key idea.** “The average of the first  $k$  scores is an integer” means “the sum of the first  $k$  scores is divisible by  $k$ .” Rewriting averages as divisibility of sums turns the problem into remainders, and it is easiest to work *backwards* from the full total.

**Step 1. The total.**  $71 + 76 + 80 + 82 + 91 = 400$ , which is divisible by 5, as it must be.

**Step 2. Use the first four scores.** Their sum is  $400 - (\text{last score})$ , and it must be divisible by 4. Since 400 is divisible by 4, the last score must be divisible by 4 too. Among the scores only  $76 = 4 \cdot 19$  and  $80 = 4 \cdot 20$  qualify.

**Step 3. Use the first three scores.** The first three sum to  $(\text{sum of first four}) - (\text{fourth score})$  and must be divisible by 3.

- If 76 were last, the first four would sum to  $400 - 76 = 324$ , a multiple of 3. Then the fourth score would itself have to be a multiple of 3. But none of 71, 80, 82, 91 is (their remainders on division by 3 are 2, 2, 1, 1). Impossible.
- If 80 is last, the first four sum to 320, which leaves remainder 2 on division by 3. The fourth score must also leave remainder 2: that is 71. The first three are then 76, 82, 91, with sum  $249 = 3 \cdot 83$ .  
✓

**Step 4. Check the first two.**  $76 + 82 = 158$  is even, so the order 76, 82, 91, 71, 80 works: the running sums 76, 158, 249, 320, 400 are divisible by 1, 2, 3, 4, 5.

So the last score was 80.

**Watch out.** Step 2 leaves *two* candidates. Stopping there and taking the first one, 76, gives choice (B); Step 3 is what rules it out.

**Answer: (C) 80**

## 6 Algebra

Source: 2009 AMC 10B #15

When a bucket is two-thirds full of water, the bucket and water weigh  $a$  kilograms. When the bucket is one-half full of water the total weight is  $b$  kilograms. In terms of  $a$  and  $b$ , what is the total weight in kilograms when the bucket is full of water?

- (A)  $\frac{2}{3}a + \frac{1}{3}b$       (B)  $\frac{3}{2}a - \frac{1}{2}b$       (C)  $\frac{3}{2}a + b$       (D)  $\frac{3}{2}a + 2b$       (E)  $3a - 2b$

**Key idea.** The total weight is made of two unknown pieces — the empty bucket and a full load of water — so give each a letter. Two facts, two unknowns: subtracting the equations kills the bucket.

**Step 1. Set up.** Let  $x$  be the weight of the empty bucket and  $y$  the weight of a full bucket's worth of water. Then

$$x + \frac{2}{3}y = a, \quad x + \frac{1}{2}y = b.$$

**Step 2. Subtract.** The bucket cancels:  $\frac{2}{3}y - \frac{1}{2}y = a - b$ . Since  $\frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$ , this says  $\frac{1}{6}y = a - b$ , so  $y = 6a - 6b$ .

**Step 3. Find the bucket.** From the second equation,  $x = b - \frac{1}{2}y = b - (3a - 3b) = 4b - 3a$ .

**Step 4. Full bucket.**

$$x + y = (4b - 3a) + (6a - 6b) = 3a - 2b.$$

**Another way.** Think of the weight going up in equal steps. Going from half full to two-thirds full adds  $\frac{1}{6}$  of the water and raises the weight by  $a - b$ . Going from two-thirds full to completely full adds another  $\frac{1}{3} = \frac{2}{6}$  of the water, i.e. two more such steps. So the full weight is  $a + 2(a - b) = 3a - 2b$ .

*Check with numbers:* a bucket of 1 kg holding 6 kg of water gives  $a = 5$ ,  $b = 4$ , and full weight 7. Indeed  $3(5) - 2(4) = 7$ , while the other four choices give  $\frac{14}{3}$ ,  $\frac{11}{2}$ ,  $\frac{23}{2}$  and  $\frac{31}{2}$ .

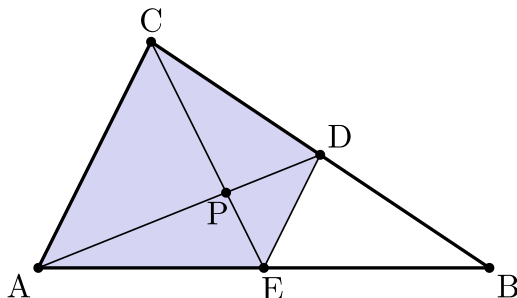
**Watch out.** It is tempting to scale  $a$  up by  $\frac{3}{2}$ , as if the bucket weighed nothing — but that counts the bucket's own weight  $1\frac{1}{2}$  times. Choices (B), (C) and (D) all start from  $\frac{3}{2}a$ ; the numerical check above shows none of them is right.

**Answer: (E)  $3a - 2b$**

## 7 Geometry

Source: 2013 AMC 10B #16

In triangle  $\triangle ABC$ , medians  $AD$  and  $CE$  intersect at  $P$ ,  $PE = 1.5$ ,  $PD = 2$ , and  $DE = 2.5$ . What is the area of  $AEDC$ ?



- (A) 13                      (B) 13.5                      (C) 14                      (D) 14.5                      (E) 15

**Key idea.** The three given lengths 1.5, 2, 2.5 are  $\frac{1}{2}$  of 3, 4, 5 — a Pythagorean triple. Whenever a triangle's sides fit  $a^2 + b^2 = c^2$ , look for a right angle; here it makes the two medians perpendicular, and a quadrilateral with perpendicular diagonals has an easy area.

**Step 1. A right angle at  $P$ .** In  $\triangle PDE$ ,

$$PE^2 + PD^2 = 1.5^2 + 2^2 = 2.25 + 4 = 6.25 = 2.5^2 = DE^2,$$

so by the converse of the Pythagorean theorem,  $\angle DPE = 90^\circ$ . Hence the medians  $AD$  and  $CE$  are perpendicular.

**Step 2. Full lengths of the medians.**  $P$  is the centroid, which divides each median in the ratio 2 : 1, with the longer part next to the vertex. So  $CP = 2 \cdot PE = 3$  and  $AP = 2 \cdot PD = 4$ . Thus

$$CE = 3 + 1.5 = 4.5, \quad AD = 4 + 2 = 6.$$

**Step 3. The area.** Cut  $AEDC$  along the diagonal  $AD$  into  $\triangle ACD$  and  $\triangle AED$ . Because  $CE \perp AD$ , the heights of these triangles to the base  $AD$  are  $CP = 3$  and  $EP = 1.5$ . So

$$[AEDC] = \frac{1}{2} \cdot 6 \cdot 3 + \frac{1}{2} \cdot 6 \cdot 1.5 = 9 + 4.5 = 13.5.$$

This is the general rule: a quadrilateral with perpendicular diagonals  $d_1, d_2$  has area  $\frac{1}{2}d_1d_2 = \frac{1}{2} \cdot 6 \cdot 4.5$ .

**Answer: (B) 13.5**

## 8 Number Theory

Source: 2007 AMC 10A #17

Suppose that  $m$  and  $n$  are positive integers such that  $75m = n^3$ . What is the minimum possible value of  $m + n$ ?

- (A) 15                      (B) 30                      (C) 50                      (D) 60                      (E) 5700

**Key idea.** “ $n^3$ ” means a perfect cube, and a number is a cube exactly when every prime in its factorisation appears a multiple of 3 times. So factor 75 and ask what  $m$  must supply.

**Step 1. Factor.**  $75 = 3 \cdot 5^2$ , so  $75m = 3^1 \cdot 5^2 \cdot m$ .

**Step 2. What  $m$  must contain.** For  $3^1 \cdot 5^2 \cdot m$  to be a cube, the exponent of 3 must reach a multiple of 3, so  $m$  needs at least two more 3's; the exponent of 5 needs at least one more 5. Hence  $m$  is a multiple of  $3^2 \cdot 5 = 45$ .

**Step 3. The smallest such  $m$  works.** With  $m = 45$ :  $75 \cdot 45 = 3375 = 3^3 \cdot 5^3 = 15^3$ , so  $n = 15$  and  $m + n = 60$ .

**Step 4. Nothing smaller is possible.** Since  $n^3$  is divisible by 3 and by 5, so is  $n$ ; thus  $n = 15k$  for some positive integer  $k$ . Then  $m = \frac{(15k)^3}{75} = 45k^3$ , so

$$m + n = 45k^3 + 15k,$$

which grows with  $k$ . Its smallest value is at  $k = 1$ :  $45 + 15 = 60$ .

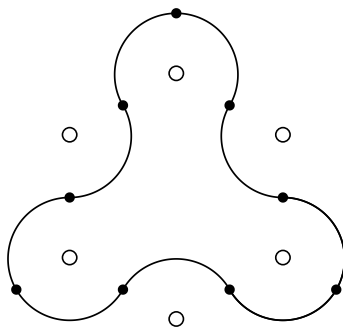
**Watch out.** A quick way to make  $75m$  a cube is  $m = 75^2$ , so that  $75m = 75^3$ . That gives  $n = 75$  and  $m + n = 5625 + 75 = 5700$ , choice (E) — a valid pair, but far from the smallest. You only need to supply the *missing* prime factors.

**Answer: (D) 60**

9 Geometry

Source: 2012 AMC 10A #18

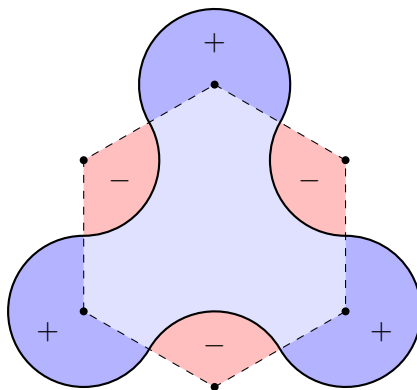
The closed curve in the figure is made up of 9 congruent circular arcs each of length  $\frac{2\pi}{3}$ , where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?



- (A)  $2\pi + 6$  (B)  $2\pi + 4\sqrt{3}$  (C)  $3\pi + 4$   
 (D)  $2\pi + 3\sqrt{3} + 2$  (E)  $\pi + 6\sqrt{3}$

**Key idea.** The six small circles in the figure are the hexagon's vertices, so draw the hexagon in. The curve then bulges outward around three vertices and dips inward at the other three, so the area is “hexagon + bulges – dips” — and every bulge and dip is a sector of a circle.

**Step 1. The radius.** An arc of radius  $r$  and angle  $120^\circ$  has length  $\frac{1}{3} \cdot 2\pi r$ . Setting  $\frac{2\pi r}{3} = \frac{2\pi}{3}$  gives  $r = 1$ . Adjacent vertices are 2 apart, so neighbouring circles of radius 1 meet exactly at the midpoint of each side; the curve passes through all six midpoints.



**Step 2. The hexagon.** It consists of six equilateral triangles of side 2, each of area  $\frac{\sqrt{3}}{4} \cdot 2^2 = \sqrt{3}$ . So the hexagon has area  $6\sqrt{3}$ .

**Step 3. The bulges (+).** At the top, lower-left and lower-right vertices, the hexagon's  $120^\circ$  corner is inside the hexagon and the curve runs around the other  $360^\circ - 120^\circ = 240^\circ$ , outside it (two of the nine arcs). Each bulge is a  $240^\circ$  sector of a unit circle, with area  $\frac{2}{3}\pi$ . Three of them add  $2\pi$ .

**Step 4. The dips (–).** At the other three vertices a single  $120^\circ$  arc cuts into the hexagon, removing a  $120^\circ$  sector of area  $\frac{1}{3}\pi$ . Three of them remove  $\pi$ .

**Step 5. Total.**  $6\sqrt{3} + 2\pi - \pi = \pi + 6\sqrt{3}$ . (Check:  $3 \cdot 2 + 3 \cdot 1 = 9$  arcs, as stated.)

**Watch out.** The dips *subtract* area. Adding all nine  $120^\circ$  sectors to the hexagon gives  $3\pi + 6\sqrt{3}$ , which is not a choice. Estimating will not rescue you either: choices (B), (C), (D), (E) all lie between 13.2 and 13.6.

**Answer: (E)**  $\pi + 6\sqrt{3}$

## 10 Algebra

Source: 2004 AMC 10B #19

In the sequence 2001, 2002, 2003,  $\dots$ , each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is  $2001 + 2002 - 2003 = 2000$ . What is the 2004th term in this sequence?

- (A)  $-2004$       (B)  $-2$       (C)  $0$       (D)  $4003$       (E)  $6007$

**Key idea.** Nobody computes 2004 terms. When a question asks for a far-off term, write out the first several and look for a pattern — then prove it from the rule so you can trust it.

**Step 1. Compute a few terms.** Call the terms  $a_1, a_2, \dots$ ; the rule is  $a_{k+1} = a_{k-2} + a_{k-1} - a_k$ .

$$a_5 = 2002 + 2003 - 2000 = 2005, \quad a_6 = 2003 + 2000 - 2005 = 1998, \quad a_7 = 2000 + 2005 - 1998 = 2007.$$

So the sequence runs 2001, 2002, 2003, 2000, 2005, 1998, 2007,  $\dots$ . The odd-numbered terms go up by 2; the even-numbered terms 2002, 2000, 1998,  $\dots$  go down by 2.

**Step 2. Why the pattern lasts.** Subtract  $a_{k-1}$  from both sides of the rule:

$$a_{k+1} - a_{k-1} = a_{k-2} - a_k = -(a_k - a_{k-2}).$$

So each “jump of two places” is the negative of the previous one. The first is  $a_3 - a_1 = +2$ , so they go  $+2, -2, +2, -2, \dots$  — that is,  $a_4 - a_2 = -2$ ,  $a_6 - a_4 = -2$ , and so on for every even-numbered term.

**Step 3. The 2004th term.** The even-numbered terms are  $a_2 = 2002$  and then drop by 2 each time. Since  $a_{2004}$  is the 1002nd even-numbered term, it comes after 1001 drops:

$$a_{2004} = 2002 - 2 \cdot 1001 = 2002 - 2002 = 0.$$

A neat way to see it:  $a_{2j} = 2004 - 2j$ , which fits  $a_2 = 2002$ ,  $a_4 = 2000$ ,  $a_6 = 1998$ , and gives  $a_{2004} = 0$ .

**Watch out.** Counting the drops is an off-by-one trap. The 1002nd even term is reached after 1001 drops, not 1002. Using 1002 gives  $2002 - 2004 = -2$ , choice (B).

**Answer: (C)**  $0$