
AMC 10

SET 16

HALF-LENGTH PRACTICE

SOLUTIONS

[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	D	2001 AMC 10 #10	Algebra
2	C	2006 AMC 10A #12	Geometry
3	A	2003 AMC 10A #13	Algebra
4	C	2019 AMC 10B #14	Number Theory
5	C	2011 AMC 10A #15	Algebra
6	B	2008 AMC 10A #16	Geometry
7	D	2019 AMC 10A #17	Counting & Probability
8	D	2018 AMC 10A #18	Number Theory
9	D	2014 AMC 10B #19	Counting & Probability
10	D	2010 AMC 10A #20	Geometry

1 Algebra

Source: 2001 AMC 10 #10

If x , y , and z are positive with $xy = 24$, $xz = 48$, and $yz = 72$, then $x + y + z$ is

- (A) 18 (B) 19 (C) 20 (D) 22 (E) 24

Key idea. Each equation uses two of the three letters, so multiplying all three equations together gives every letter twice: $(xyz)^2$.

Step 1. Multiply: $(xy)(xz)(yz) = (xyz)^2 = 24 \cdot 48 \cdot 72$. In primes, $24 = 2^3 \cdot 3$, $48 = 2^4 \cdot 3$ and $72 = 2^3 \cdot 3^2$, so the product is $2^{10} \cdot 3^4$. Since the numbers are positive, $xyz = 2^5 \cdot 3^2 = 288$.

Step 2. Divide xyz by each given product to free one letter:

$$x = \frac{288}{yz} = \frac{288}{72} = 4, \quad y = \frac{288}{xz} = \frac{288}{48} = 6, \quad z = \frac{288}{xy} = \frac{288}{24} = 12.$$

Check: $4 \cdot 6 = 24$, $4 \cdot 12 = 48$, $6 \cdot 12 = 72$.

Step 3. $x + y + z = 4 + 6 + 12 = 22$.

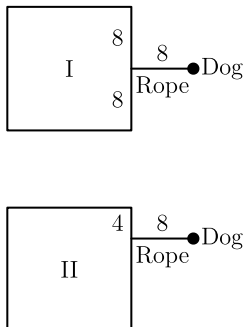
Another way. Divide two of the equations so that one letter cancels. Dividing $xz = 48$ by $xy = 24$ gives $\frac{z}{y} = 2$, so $z = 2y$. Then $yz = 72$ becomes $y \cdot 2y = 2y^2 = 72$, so $y^2 = 36$ and, as y is positive, $y = 6$. Hence $z = 12$ and $x = \frac{24}{6} = 4$, the same values as before.

Answer: (D) 22

2 Geometry

Source: 2006 AMC 10A #12

Rolly wishes to secure his dog with an 8-foot rope to a square shed that is 16 feet on each side. His preliminary drawings are shown.



Which of these arrangements gives the dog the greater area to roam, and by how many square feet?

- (A) I, by 8π (B) I, by 6π (C) II, by 4π (D) II, by 8π (E) II, by 10π

Key idea. Picture where the rope can reach. In front of the wall the dog always sweeps a half-disk; the only real question is whether the rope is long enough to bend around a corner of the shed — and the labels 8 and 4 in the drawings point straight at the corners.

Step 1. Arrangement I. The rope is tied at the middle of a 16-foot side. In front of that wall the dog sweeps a half-disk of radius 8, with area $\frac{1}{2}\pi \cdot 8^2 = 32\pi$. Each corner is exactly 8 feet from the knot, so the rope just reaches the corners with nothing left to wrap around them. Total: 32π .

Step 2. Arrangement II. The knot is 4 feet from one corner and 12 feet from the other. In front of the wall the dog again sweeps a half-disk of area 32π . The far corner is out of reach, but the near corner is reached with $8 - 4 = 4$ feet of rope to spare.

Pivoting at that corner, the 4-foot piece can swing from straight along the line of the wall until it lies flat against the next wall — a quarter-turn. It sweeps a quarter-disk of radius 4, with area $\frac{1}{4}\pi \cdot 4^2 = 4\pi$, which does not overlap the half-disk. Total: $32\pi + 4\pi = 36\pi$.

Step 3. Arrangement II is larger, by $36\pi - 32\pi = 4\pi$ square feet.

Watch out. The shed blocks the swing past the corner, so the extra region is a *quarter*-disk. Treating it as a half-disk gives $\frac{1}{2}\pi \cdot 4^2 = 8\pi$ and the answer “II, by 8π ”, choice (D).

Answer: (C) II, by 4π

3 Algebra

Source: 2003 AMC 10A #13

The sum of three numbers is 20. The first is 4 times the sum of the other two. The second is seven times the third. What is the product of all three?

- (A) 28 (B) 40 (C) 100 (D) 400 (E) 800

Key idea. Turn each sentence into an equation. Both the total and the first condition involve “the other two” together, so treat $b + c$ as a single block and solve for it first.

Call the numbers a, b, c in order. The three sentences say

$$a + b + c = 20, \quad a = 4(b + c), \quad b = 7c.$$

Step 1. Substitute $a = 4(b + c)$ into the sum: $4(b + c) + (b + c) = 20$, that is, $5(b + c) = 20$. So $b + c = 4$ and $a = 4 \cdot 4 = 16$.

Step 2. Now use $b = 7c$ in $b + c = 4$: $7c + c = 8c = 4$, so $c = \frac{1}{2}$ and $b = 7 \cdot \frac{1}{2} = \frac{7}{2}$.

Check all three conditions: $16 + \frac{7}{2} + \frac{1}{2} = 16 + 4 = 20$; $16 = 4 \cdot 4$; and $\frac{7}{2} = 7 \cdot \frac{1}{2}$. All hold.

Step 3. The product is

$$16 \cdot \frac{7}{2} \cdot \frac{1}{2} = 16 \cdot \frac{7}{4} = 28.$$

Watch out. Nothing says the numbers are whole numbers. When c comes out as $\frac{1}{2}$, that is correct — do not go back looking for an “integer” mistake, and do not round.

Another way. Think in shares. The first number is 4 times the other two together, so the total 20 splits in the ratio 4 : 1: the first is $\frac{4}{5} \cdot 20 = 16$ and the other two make 4. In the same way $b : c = 7 : 1$ splits that 4 into $\frac{7}{8} \cdot 4 = \frac{7}{2}$ and $\frac{1}{8} \cdot 4 = \frac{1}{2}$.

Answer: (A) 28

4 Number Theory

Source: 2019 AMC 10B #14

The base-ten representation for $19!$ is $121,6T5,100,40M,832,H00$, where T , M , and H denote digits that are not given. What is $T + M + H$?

- (A) 3 (B) 8 (C) 12 (D) 14 (E) 17

Key idea. We cannot multiply out $19!$ by hand, but we know many numbers that divide it. Pick divisors whose tests read off digits: 1000 (last three digits), 9 (digit sum) and 11 (alternating digit sum). Three unknowns, three tests.

Step 1. Divisibility by 1000. The product $19!$ contains 5, 10 and 15, giving three factors of 5, and far more than three factors of 2. So $1000 = 2^3 \cdot 5^3$ divides $19!$, its last three digits are 000, and $H = 0$.

Step 2. Divisibility by 9. The known digits add to $1+2+1+6+5+1+0+0+4+0+8+3+2+0+0 = 33$. So $33 + T + M$ is a multiple of 9. As $0 \leq T + M \leq 18$, the sum lies between 33 and 51, so it is 36 or 45: $T + M = 3$ or $T + M = 12$.

Step 3. Divisibility by 11. Starting from the units digit, add the digits in the 1st, 3rd, 5th, ... places: $0 + H + 3 + M + 4 + 0 + 5 + 6 + 2 = 20 + M$. The digits in the 2nd, 4th, ... places give $0 + 2 + 8 + 0 + 0 + 1 + T + 1 + 1 = 13 + T$. Their difference $(20 + M) - (13 + T) = 7 + M - T$ must be a multiple of 11. Since it lies between -2 and 16 , it is 0 or 11: $T - M = 7$ or $T - M = -4$.

Step 4. Combine. $T + M$ and $T - M$ have the same parity. $T + M = 3$ with $T - M = 7$ gives $M = -2$, impossible. $T + M = 12$ with $T - M = -4$ gives $T = 4$, $M = 8$. Indeed $19! = 121,645,100,408,832,000$.

So $T + M + H = 4 + 8 + 0 = 12$.

Watch out. The digit-sum test alone leaves two possibilities. Guessing the smaller one, $T + M = 3$, gives $3 + 0 = 3$, choice (A); you need the test for 11 to decide.

Answer: (C) 12

5 Algebra

Source: 2011 AMC 10A #15

Roy bought a new battery-gasoline hybrid car. On a trip the car ran exclusively on its battery for the first 40 miles, then ran exclusively on gasoline for the rest of the trip, using gasoline at a rate of 0.02 gallons per mile. On the whole trip he averaged 55 miles per gallon. How long was the trip in miles?

- (A) 140 (B) 240 (C) 440 (D) 640 (E) 840

Key idea. “Averaged 55 miles per gallon” means *total miles* divided by *total gallons*. Only the gasoline miles use any gallons, so name that unknown distance and write this one fraction.

Step 1. Set up. Let x be the number of miles driven on gasoline. The whole trip is $40 + x$ miles, and the gasoline used is $0.02x$ gallons. So

$$\frac{40 + x}{0.02x} = 55.$$

Step 2. Solve. Multiply both sides by $0.02x$: $40 + x = 55 \cdot 0.02x = 1.1x$. Subtract x : $40 = 0.1x$, so $x = 400$.

Step 3. Answer the question asked. The trip is $40 + 400 = 440$ miles. Check: the car burns $0.02 \cdot 400 = 8$ gallons, and $440 \div 8 = 55$ miles per gallon.

It helps to see why this makes sense. On gasoline alone the car gets $1 \div 0.02 = 50$ miles per gallon; the 40 free battery miles pull the average up to 55. The longer the gasoline part, the closer the average falls back to 50 — so there is just one trip length that gives exactly 55.

Watch out. $x = 400$ is only the gasoline part of the trip. The question asks for the whole trip, so add the 40 battery miles back.

Another way. Test the choices: for a trip of T miles the gallons are $0.02(T - 40)$. For $T = 440$: $0.02 \cdot 400 = 8$ and $440 \div 8 = 55$. For $T = 240$: $0.02 \cdot 200 = 4$ gallons, giving 60 — too high.

Answer: (C) 440

6 Geometry

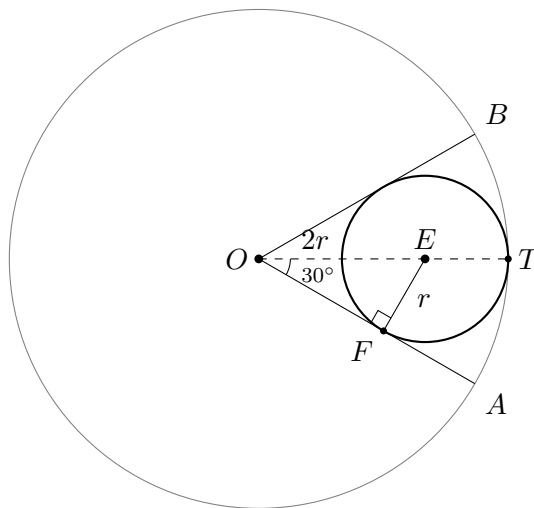
Source: 2008 AMC 10A #16

Points A and B lie on a circle centered at O , and $\angle AOB = 60^\circ$. A second circle is internally tangent to the first and tangent to both OA and OB . What is the ratio of the area of the smaller circle to that of the larger circle?

- (A) $\frac{1}{16}$ (B) $\frac{1}{9}$ (C) $\frac{1}{8}$ (D) $\frac{1}{6}$ (E) $\frac{1}{4}$

Key idea. A circle tangent to both sides of an angle has its centre on the angle bisector. Half of 60° is 30° , and a radius drawn to a tangent point makes a right angle — so a 30-60-90 triangle appears and ties r to the distance from O .

Let R and r be the radii of the large and small circles, and let E be the centre of the small circle.



Step 1. E is at distance r from both OA and OB , so it lies on the bisector of $\angle AOB$, and $\angle EOA = 30^\circ$.

Step 2. Let F be the point where the small circle touches OA . The radius $EF = r$ is perpendicular to OA . In right triangle OFE the angle at O is 30° , and the side opposite it is half the hypotenuse. So $OE = 2 \cdot EF = 2r$.

Step 3. The two circles touch at a point T on the line through their centres, so O, E, T are in a line and $OT = OE + ET$. That is, $R = 2r + r = 3r$.

Step 4. The ratio of areas is the square of the ratio of radii:

$$\frac{\pi r^2}{\pi R^2} = \left(\frac{r}{3r}\right)^2 = \frac{1}{9}.$$

Watch out. The large radius reaches past E by one more small radius. Writing $OE = R$ instead of $OE = R - r$ gives $R = 2r$ and a ratio of $\frac{1}{4}$, choice (E). And remember to square: the radius ratio $\frac{1}{3}$ is not the area ratio.

Answer: (B) $\frac{1}{9}$
7 Counting & Probability

Source: 2019 AMC 10A #17

A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

- (A) 24 (B) 288 (C) 312 (D) 1,260 (E) 40,320

Key idea. “One cube will be left out” invites a split into cases by the colour of the missing cube. In each case we arrange 8 cubes in which colours repeat — a standard “arrangements with repeated letters” count.

Step 1. The counting rule. If 8 cubes were all different, there would be $8! = 40,320$ orders. Cubes of the same colour can be swapped without changing the tower, so we divide by the number of ways to shuffle each colour among itself. For p red, q blue and s green cubes the count is $\frac{8!}{p!q!s!}$.

Step 2. Three cases.

- Red left out (1 red, 3 blue, 4 green): $\frac{40,320}{1 \cdot 6 \cdot 24} = \frac{40,320}{144} = 280$.
- Blue left out (2 red, 2 blue, 4 green): $\frac{40,320}{2 \cdot 2 \cdot 24} = \frac{40,320}{96} = 420$.
- Green left out (2 red, 3 blue, 3 green): $\frac{40,320}{2 \cdot 6 \cdot 6} = \frac{40,320}{72} = 560$.

Step 3. Towers from different cases have different colour counts, so none is counted twice. The total is $280 + 420 + 560 = 1260$.

Watch out. Using $8! = 40,320$ treats the two red cubes (and the blues, and the greens) as different from each other. The cubes of one colour are interchangeable, so (E) overcounts.

Another way. Put the left-out cube on top of the tower. This turns each height-8 tower into an arrangement of all 9 cubes, and removing the top cube undoes it. So we just count towers of all 9 cubes: $\frac{9!}{2!3!4!} = \frac{362,880}{288} = 1260$.

Answer: (D) 1,260

8 Number Theory

Source: 2018 AMC 10A #18

How many nonnegative integers can be written in the form

$$\begin{aligned} & a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 \\ & + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 \\ & + a_1 \cdot 3^1 + a_0 \cdot 3^0, \end{aligned}$$

where $a_i \in \{-1, 0, 1\}$ for $0 \leq i \leq 7$?

- (A) 512 (B) 729 (C) 1094 (D) 3281 (E) 59,048

Key idea. Coefficients $-1, 0, 1$ are just the base-3 digits $0, 1, 2$ shifted down by 1. Adding 1 to every coefficient turns the strange sum into an ordinary base-3 number — and base-3 numbers are easy to count.

Call the value of the sum N .

Step 1. Shift the coefficients. Put $d_i = a_i + 1$, so each d_i is 0, 1 or 2. Then

$$d_7 \cdot 3^7 + \cdots + d_0 \cdot 3^0 = N + (3^7 + 3^6 + \cdots + 3^0) = N + 3280,$$

because $1 + 3 + \cdots + 3^7 = \frac{3^8 - 1}{2} = \frac{6561 - 1}{2} = 3280$.

Step 2. Read it as base 3. The left side is the number written $d_7 d_6 \dots d_0$ in base 3 (leading zeros allowed). Every integer from 0 to $3^8 - 1 = 6560$ has exactly one such 8-digit base-3 form, and nothing else does. So $N + 3280$ takes each value $0, 1, \dots, 6560$ exactly once, and N itself takes each integer from -3280 to 3280 exactly once.

Step 3. Count. The nonnegative values are $0, 1, 2, \dots, 3280$: that is 3281 integers.

Watch out. There are *eight* coefficients, a_0 through a_7 . Working with seven of them gives the values -1093 to 1093 and 1094 nonnegative ones — choice (C).

Another way. Step 2 shows that the $3^8 = 6561$ choices of coefficients give 6561 different values. Swapping every 1 with -1 negates the value, so positive and negative values pair up. Exactly one value is 0, so $\frac{6561 - 1}{2} = 3280$ are positive; adding 0 gives 3281.

Answer: (D) 3281

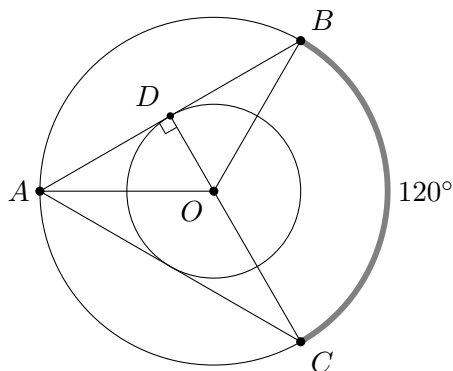
9 Counting & Probability

Source: 2014 AMC 10B #19

Two concentric circles have radii 1 and 2. Two points on the outer circle are chosen independently and uniformly at random. What is the probability that the chord joining the two points intersects the inner circle?

- (A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{2 - \sqrt{2}}{2}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

Key idea. By symmetry only the position of the second point relative to the first matters, so fix the first point A . Then find the borderline chords — the ones from A that just touch the inner circle. Radii 1 and 2 in a right triangle signal a 30-60-90 triangle.



Step 1. The tangent chord. Draw the chord AB that touches the inner circle at D . Then $OD \perp AB$, with $OD = 1$ and $OA = 2$. A right triangle whose leg is half its hypotenuse has a 30° angle opposite that leg, so $\angle OAD = 30^\circ$.

Step 2. Find arc BC . Triangle OAB is isosceles ($OA = OB = 2$), so $\angle OBA = 30^\circ$ too, and $\angle AOB = 180^\circ - 30^\circ - 30^\circ = 120^\circ$. The other tangent chord AC gives $\angle AOC = 120^\circ$ on the other side. The arc BC not containing A is therefore $360^\circ - 120^\circ - 120^\circ = 120^\circ$ (shaded).

Step 3. Which chords cross? For any chord AP , the perpendicular from O meets it at its midpoint, so by Pythagoras

$$(\text{distance from } O \text{ to } AP)^2 = 2^2 - \left(\frac{1}{2}AP\right)^2.$$

So the longer the chord, the closer it passes to O . The tangent chords AB and AC pass at distance exactly 1. As P moves from A around to the point opposite A , the chord AP only gets longer; hence AP passes closer than 1 to O (and so crosses the inner circle) exactly when it is longer than AB — that is, exactly when P lies on the shaded arc BC .

Step 4. The second point is uniform on the circle, so the probability is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$.

Watch out. The answer is not a ratio of areas. The inner disk has $\frac{1}{4}$ of the outer disk's area, but that leads to choice (B), which is wrong: the random points live on the circle, so we must measure arcs.

Answer: (D) $\frac{1}{3}$

10 Geometry

Source: 2010 AMC 10A #20

A fly trapped inside a cubical box with side length 1 meter decides to relieve its boredom by visiting each corner of the box. It will begin and end in the same corner and visit each of the other corners exactly once. To get from a corner to any other corner, it will either fly or crawl in a straight line. What is the maximum possible length, in meters, of its path?

- (A) $4 + 4\sqrt{2}$ (B) $2 + 4\sqrt{2} + 2\sqrt{3}$ (C) $2 + 3\sqrt{2} + 3\sqrt{3}$
 (D) $4\sqrt{2} + 4\sqrt{3}$ (E) $3\sqrt{2} + 5\sqrt{3}$

Key idea. Between two corners there are only three possible distances: 1, $\sqrt{2}$ and $\sqrt{3}$. To make the path long we want many $\sqrt{3}$ moves — so count how many such moves even exist. A maximum needs two halves: a bound, and a path that reaches it.

Step 1. The possible moves. Put the cube at coordinates 0 and 1, so each corner is a triple like $(0, 1, 1)$. Two corners that differ in k coordinates are $\sqrt{k}$ apart: 1 (an edge), $\sqrt{2}$ (a face diagonal) or $\sqrt{3}$ (a space diagonal). The path has 8 moves: it visits 8 corners and returns to the start.

Step 2. The bound. Each corner has exactly one opposite corner, so the cube has only 4 space diagonals. The path never repeats a move (it visits each corner once), so at most 4 of its 8 moves have length $\sqrt{3}$; every other move is at most $\sqrt{2}$. If the path uses $k \leq 4$ space diagonals, its length is at most $k\sqrt{3} + (8 - k)\sqrt{2} \leq 4\sqrt{3} + 4\sqrt{2}$, since $\sqrt{3} > \sqrt{2}$.

Step 3. A path that achieves it. Alternate “change all three coordinates” with “change two coordinates”:

$$\begin{aligned} (0, 0, 0) &\rightarrow (1, 1, 1) \rightarrow (0, 0, 1) \rightarrow (1, 1, 0) \rightarrow (0, 1, 1) \\ &\rightarrow (1, 0, 0) \rightarrow (0, 1, 0) \rightarrow (1, 0, 1) \rightarrow (0, 0, 0). \end{aligned}$$

All 8 corners appear once, and the moves alternate $\sqrt{3}, \sqrt{2}, \sqrt{3}, \sqrt{2}, \dots$ — four of each.

So the maximum is $4\sqrt{2} + 4\sqrt{3} \approx 12.59$ meters.

Watch out. Choice (E), $3\sqrt{2} + 5\sqrt{3} \approx 12.90$, is larger, but it needs five space diagonals, and a cube has only four.

Answer: (D) $4\sqrt{2} + 4\sqrt{3}$