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# AMC 10

# SET 18

HALF-LENGTH PRACTICE

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# SOLUTIONS

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[Answer key](#) • [Sources](#) • [Step-by-step solutions](#)

All 10 problems come from positions 10–20 of past AMC 10 contests (2000–2019) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

## Answer Key and Sources

| No. | Ans. | Source           | Topic                  |
|-----|------|------------------|------------------------|
| 1   | D    | 2004 AMC 10B #10 | Algebra                |
| 2   | A    | 2013 AMC 10A #11 | Counting & Probability |
| 3   | D    | 2001 AMC 10 #12  | Number Theory          |
| 4   | E    | 2006 AMC 10B #13 | Algebra                |
| 5   | A    | 2007 AMC 10A #14 | Geometry               |
| 6   | C    | 2003 AMC 10A #16 | Number Theory          |
| 7   | D    | 2015 AMC 10A #17 | Geometry               |
| 8   | D    | 2018 AMC 10B #18 | Counting & Probability |
| 9   | B    | 2019 AMC 10A #19 | Algebra                |
| 10  | D    | 2004 AMC 10A #20 | Geometry               |

**1** Algebra

Source: 2004 AMC 10B #10

A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

- (A) 5                      (B) 8                      (C) 9                      (D) 10                      (E) 11

**Key idea.** The rows hold  $1, 3, 5, 7, \dots$  cans — the odd numbers — and the sum of the first  $n$  odd numbers is always  $n^2$ .

Count the cans row by row and keep a running total:

$$1 = 1^2, \quad 1 + 3 = 4 = 2^2, \quad 1 + 3 + 5 = 9 = 3^2, \quad 1 + 3 + 5 + 7 = 16 = 4^2.$$

Each new odd number completes the next perfect square. To see why, picture the cans rearranged as an  $n \times n$  square; wrapping an L-shaped border of  $n + n + 1 = 2n + 1$  cans around two sides makes an  $(n + 1) \times (n + 1)$  square, and  $2n + 1$  is exactly the next odd number. So a display with  $n$  rows holds exactly  $n^2$  cans.

We need  $n^2 = 100$ , so  $n = 10$ . The neighbouring choices fail: 9 rows hold only 81 cans and 11 rows would need 121. As a check, the bottom row has  $2 \cdot 10 - 1 = 19$  cans, and

$$1 + 3 + 5 + \dots + 19 = \frac{10(1 + 19)}{2} = 100.$$

**Answer: (D) 10****2** Counting & Probability

Source: 2013 AMC 10A #11

A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

- (A) 10                      (B) 12                      (C) 15                      (D) 18                      (E) 25

**Key idea.** The number 10 is a clue about the size of the council. Use it to find how many members there are, then count the three-person committees directly.

**Step 1.** Let the council have  $x$  members. To choose 2 of them, there are  $x$  choices for one person and  $x - 1$  for the other, but this counts every pair twice (once in each order). So the number of ways to choose 2 of them (order does not matter) is

$$\binom{x}{2} = \frac{x(x-1)}{2}.$$

We are told this equals 10, so  $x(x-1) = 20$ .

**Step 2.** Solve:  $x^2 - x - 20 = 0$  factors as  $(x - 5)(x + 4) = 0$ . A council cannot have  $-4$  members, so  $x = 5$ . (Check:  $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$ .)

**Step 3.** Since students may serve on both committees, the planning committee is chosen from all 5 members:

$$\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10.$$

Notice why the two counts agree: choosing the 3 people who *are* on the committee is the same as choosing the 2 people who are *not*, so  $\binom{5}{3} = \binom{5}{2}$ .

**Answer: (A) 10**

### 3 Number Theory

Source: 2001 AMC 10 #12

Suppose that  $n$  is the product of three consecutive integers and that  $n$  is divisible by 7. Which of the following is not necessarily a divisor of  $n$ ?

- (A) 6                      (B) 14                      (C) 21                      (D) 28                      (E) 42

**Key idea.** Ask which prime factors are *guaranteed*. Every choice is built from 2, 3 and 7, so find how many of each  $n$  must contain — then look for a choice that needs more.

**Step 1: what is guaranteed.** Among any two consecutive integers one is even, so  $n$  has at least one factor of 2. Among any three consecutive integers one is a multiple of 3, so  $n$  has a factor of 3. We are told  $7 \mid n$ . Since 2, 3, 7 are different primes,  $n$  is a multiple of  $2 \cdot 3 \cdot 7 = 42$ .

**Step 2: check the choices.** Each of  $6 = 2 \cdot 3$ ,  $14 = 2 \cdot 7$ ,  $21 = 3 \cdot 7$  and  $42 = 2 \cdot 3 \cdot 7$  divides 42, so each always divides  $n$ . But  $28 = 2^2 \cdot 7$  needs *two* factors of 2, and only one is guaranteed.

**Step 3: a counterexample.** Take  $5 \cdot 6 \cdot 7 = 210$ . It is a product of three consecutive integers and is divisible by 7, but  $210 = 2 \cdot 3 \cdot 5 \cdot 7$  has only one 2, and  $210 \div 28 = 7.5$ . So 28 need not divide  $n$ .

**Watch out.** It is tempting to pick the largest number, 42 (choice E), as the “hardest” to divide. Size is not what matters: 42 uses each guaranteed prime only once, while the smaller 28 needs a second 2.

**Answer: (D) 28**

## 4 Algebra

Source: 2006 AMC 10B #13

Joe and JoAnn each bought 12 ounces of coffee in a 16-ounce cup. Joe drank 2 ounces of his coffee and then added 2 ounces of cream. JoAnn added 2 ounces of cream, stirred the coffee well, and then drank 2 ounces. What is the resulting ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee?

- (A)  $\frac{6}{7}$                       (B)  $\frac{13}{14}$                       (C) 1                      (D)  $\frac{14}{13}$                       (E)  $\frac{7}{6}$

**Key idea.** The order of the steps matters. Track only the cream: Joe drank before adding cream, so he loses none of it; JoAnn drank a well-stirred mixture, so she drank cream in proportion to how much of her cup was cream.

**Joe.** He drinks 2 ounces of pure coffee, then adds 2 ounces of cream. Nothing is drunk afterwards, so he has 2 ounces of cream.

**JoAnn.** After adding cream she has  $12 + 2 = 14$  ounces of evenly mixed liquid, of which 2 ounces is cream, so cream is  $\frac{2}{14} = \frac{1}{7}$  of every sip. She drinks 2 ounces, which contains  $2 \cdot \frac{1}{7} = \frac{2}{7}$  ounce of cream. She keeps

$$2 - \frac{2}{7} = \frac{12}{7} \text{ ounces of cream.}$$

(Equivalently, she keeps  $\frac{12}{14} = \frac{6}{7}$  of her liquid, hence  $\frac{6}{7}$  of her 2 ounces of cream.)

**Ratio.** Joe's cream to JoAnn's cream:

$$\frac{2}{12/7} = 2 \cdot \frac{7}{12} = \frac{7}{6}.$$

**Watch out.** "Both added 2 ounces of cream and both drank 2 ounces, so it's the same" gives 1 (choice C). Joe's 2 ounces were pure coffee; JoAnn's contained some cream.

**Watch out.** Writing JoAnn to Joe instead of Joe to JoAnn gives  $\frac{6}{7}$  (choice A). Read the order in the question.

**Answer: (E)**  $\frac{7}{6}$

## 5 Geometry

Source: 2007 AMC 10A #14

A triangle with side lengths in the ratio 3 : 4 : 5 is inscribed in a circle of radius 3. What is the area of the triangle?

- (A) 8.64                      (B) 12                      (C)  $5\pi$                       (D) 17.28                      (E) 18

**Key idea.** The ratio 3 : 4 : 5 means a *right* triangle, and a right triangle inscribed in a circle always has its hypotenuse as a diameter. That links the unknown size of the triangle to the radius.

**Step 1.** Let the sides be  $3x$ ,  $4x$ ,  $5x$ . Since  $(3x)^2 + (4x)^2 = 9x^2 + 16x^2 = 25x^2 = (5x)^2$ , the triangle has a right angle opposite the side  $5x$ .

**Step 2.** An inscribed angle of  $90^\circ$  cuts off a semicircle, so the hypotenuse is a diameter:

$$5x = 2 \cdot 3 = 6, \quad x = \frac{6}{5}.$$

**Step 3.** The legs are  $3x = \frac{18}{5}$  and  $4x = \frac{24}{5}$ , and they are perpendicular, so

$$\text{Area} = \frac{1}{2} \cdot \frac{18}{5} \cdot \frac{24}{5} = \frac{216}{25} = 8.64.$$

(Equivalently,  $\frac{1}{2}(3x)(4x) = 6x^2 = 6 \cdot \frac{36}{25}$ .)

**Watch out.** Forgetting the  $\frac{1}{2}$  in the triangle-area formula gives  $\frac{18}{5} \cdot \frac{24}{5} = 17.28$ , choice D — exactly double the right answer.

**Watch out.** Setting the hypotenuse equal to the *radius* ( $5x = 3$ ) instead of the diameter leads to an area of 2.16, which is not a choice; if you get that, recheck which length the hypotenuse matches.

|                         |
|-------------------------|
| <b>Answer: (A)</b> 8.64 |
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## 6 Number Theory

Source: 2003 AMC 10A #16

What is the units digit of  $13^{2003}$ ?

- (A) 1                      (B) 3                      (C) 7                      (D) 8                      (E) 9

**Key idea.** The exponent is far too big to compute, but only the last digit is asked for. Last digits of powers depend only on the last digit of the base, and they repeat in a short cycle — so find the cycle and use the remainder of the exponent.

**Step 1: reduce the base.** When multiplying, the units digit of the product depends only on the units digits of the factors. So  $13^{2003}$  has the same units digit as  $3^{2003}$ .

**Step 2: find the cycle.** Units digits of powers of 3:

$$3^1 = 3, \quad 3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 81, \quad 3^5 = 243, \dots$$

giving 3, 9, 7, 1, 3, ... . Once a units digit of 1 appears, multiplying by 3 starts the pattern again, so the digits repeat every 4 powers.

**Step 3: use the remainder.** Divide the exponent by the cycle length:

$$2003 = 4 \cdot 500 + 3.$$

So  $3^{2003} = (3^4)^{500} \cdot 3^3$ . The factor  $(3^4)^{500}$  ends in 1, so the units digit is that of  $3^3 = 27$ , namely 7.

**Watch out.** A remainder of 3 means “like  $3^3$ .” If you list the cycle starting from  $3^0 = 1$  as 1, 3, 9, 7 and take the third entry, you get 9 (choice E) — that is  $3^2$ , one step short.

**Answer: (C) 7**

## 7 Geometry

Source: 2015 AMC 10A #17

A line that passes through the origin intersects both the line  $x = 1$  and the line

$$y = 1 + \frac{\sqrt{3}}{3}x.$$

The three lines create an equilateral triangle. What is the perimeter of the triangle?

- (A)  $2\sqrt{6}$                       (B)  $2 + 2\sqrt{3}$                       (C) 6                      (D)  $3 + 2\sqrt{3}$                       (E)  $6 + \frac{\sqrt{3}}{3}$

**Key idea.** In an equilateral triangle every pair of sides meets at  $60^\circ$ . Two of the lines are known, so think in terms of the *angle* each line makes with the horizontal; that pins down the slope of the unknown line.

**Step 1: angles of the known lines.** The line  $x = 1$  is vertical: it makes  $90^\circ$  with the horizontal. The line  $y = 1 + \frac{\sqrt{3}}{3}x$  has slope  $\frac{\sqrt{3}}{3} = \tan 30^\circ$ , so it makes  $30^\circ$  with the horizontal. These two lines meet at  $90^\circ - 30^\circ = 60^\circ$ , as they should.

**Step 2: the third line.** It must make  $60^\circ$  with the vertical line, so it is tilted  $30^\circ$  from horizontal, either up or down. Tilted up, it would be parallel to the second line and make no triangle. So it is tilted down: slope  $-\frac{\sqrt{3}}{3}$ , and through the origin its equation is  $y = -\frac{\sqrt{3}}{3}x$ . Check the last angle: this line is  $30^\circ$  below horizontal and the second line is  $30^\circ$  above, so they meet at  $30^\circ + 30^\circ = 60^\circ$  too.

**Step 3: one side length.** The side on  $x = 1$  joins the points where the other two lines cross  $x = 1$ :

$$y = 1 + \frac{\sqrt{3}}{3} \quad \text{and} \quad y = -\frac{\sqrt{3}}{3}.$$

Its length is  $1 + \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = 1 + \frac{2\sqrt{3}}{3}$ .

**Step 4: perimeter.** All three sides are equal, so the perimeter is

$$3 \left( 1 + \frac{2\sqrt{3}}{3} \right) = 3 + 2\sqrt{3}.$$

(The third vertex is  $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ ; its distance to  $(1, -\frac{\sqrt{3}}{3})$  is also  $1 + \frac{2\sqrt{3}}{3}$ .)

**Answer: (D)**  $3 + 2\sqrt{3}$

## 8 Counting & Probability

Source: 2018 AMC 10B #18

Three young brother-sister pairs from different families need to take a trip in a van. These six children will occupy the second and third rows in the van, each of which has three seats. To avoid disruptions, siblings may not sit right next to each other in the same row, and no child may sit directly in front of his or her sibling. How many seating arrangements are possible for this trip?

- (A) 60                      (B) 72                      (C) 92                      (D) 96                      (E) 120

**Key idea.** With so many restrictions, first figure out what an allowed seating must *look like*. It turns out each row must contain exactly one child from each family — and after that the count is a simple product.

Label the seats in each row left, middle, right; “directly in front of” means the same position in the other row.

**Step 1: no family can share a row.** Suppose family  $X$  has both children in one row. They cannot be adjacent, so they take the left and right seats, and a child from family  $Y$  sits in the middle. The other row now holds the other  $Y$  child and both  $Z$  children. The  $Z$  siblings also must avoid being adjacent, so they take left and right — forcing the second  $Y$  child into the middle, directly behind (or in front of) the first. That is forbidden. So every row has one child from each family, and then no siblings are ever side by side.

**Step 2: the second row's families.** Decide which family sits in each seat of the second row:  $3! = 6$  ways.

**Step 3: the third row's families.** Each family must sit in a *different* position from the one it had in front. If the second row is  $X, Y, Z$ , the third row must be  $Y, Z, X$  or  $Z, X, Y$  — only 2 ways (check: the other four orders all leave some family in the same spot).

**Step 4: which sibling goes where.** For each family, either child can take the front seat:  $2^3 = 8$  ways.

Multiplying,  $6 \cdot 2 \cdot 8 = 96$ .

**Watch out.** Counting only family patterns and forgetting that each brother and sister can swap seats gives  $6 \cdot 2 = 12$ , far too small; the factor  $2^3$  is easy to miss.

**Answer: (D) 96**

## 9 Algebra

Source: 2019 AMC 10A #19

What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4) + 2019,$$

where  $x$  is a real number?

- (A) 2017                      (B) 2018                      (C) 2019                      (D) 2020                      (E) 2021

**Key idea.** Multiplying four factors out directly gives a messy quartic. Instead pair the outer factors and the inner factors: both products start  $x^2 + 5x$ , which makes the expression a quadratic in one block, and a square tells you its minimum.

**Step 1: pair the factors.** Pair 1 with 4 and 2 with 3 (each pair sums to 5):

$$(x+1)(x+4) = x^2 + 5x + 4, \quad (x+2)(x+3) = x^2 + 5x + 6.$$

**Step 2: complete the square.** Let  $u = x^2 + 5x + 5$ , the number halfway between them. Then the product is

$$(u-1)(u+1) = u^2 - 1,$$

so the whole expression equals  $u^2 - 1 + 2019 = u^2 + 2018$ .

**Step 3: minimise.** Since  $u^2 \geq 0$ , the expression is at least 2018. Equality needs  $u = 0$ , i.e.  $x^2 + 5x + 5 = 0$ . Its discriminant is  $5^2 - 4 \cdot 5 = 5 > 0$ , so real solutions exist ( $x = \frac{-5 \pm \sqrt{5}}{2}$ ). Hence 2018 is actually reached, and it is the least value.

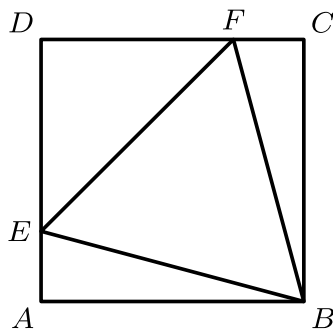
**Watch out.** It is tempting to say the product is at least 0 (it equals 0 at  $x = -1$ ), giving 2019, choice C. But the product can be negative: at  $x = -1.5$  it is  $(-0.5)(0.5)(1.5)(2.5) = -0.9375$ . Its true minimum is  $-1$ .

**Answer: (B) 2018**

10 Geometry

Source: 2004 AMC 10A #20

Points  $E$  and  $F$  are located on square  $ABCD$  so that  $\triangle BEF$  is equilateral. What is the ratio of the area of  $\triangle DEF$  to that of  $\triangle ABE$ ?



- (A)  $\frac{4}{3}$       (B)  $\frac{3}{2}$       (C)  $\sqrt{3}$       (D) 2      (E)  $1 + \sqrt{3}$

**Key idea.** Both triangles in the question are right triangles with legs along the square, so their areas come from a single unknown length. The equilateral condition,  $EF = BE$ , gives one equation for that length — and you may not even need to solve it.

**Step 1: set up.** Let the square have side 1, with  $E$  on  $\overline{AD}$  and  $F$  on  $\overline{DC}$ . Right triangles  $ABE$  and  $CBF$  have equal hypotenuses ( $BE = BF$ ) and equal legs ( $AB = CB$ ), so they are congruent and  $AE = CF$ . Hence  $DE = DF$ ; call this length  $x$ , so  $AE = 1 - x$ .

**Step 2: use  $EF = BE$ .** By the Pythagorean theorem,

$$EF^2 = x^2 + x^2 = 2x^2, \quad BE^2 = 1^2 + (1 - x)^2.$$

Setting them equal:  $2x^2 = 1 + 1 - 2x + x^2$ , so  $x^2 = 2 - 2x = 2(1 - x)$ .

**Step 3: the ratio.** Both triangles are right triangles:

$$[DEF] = \frac{1}{2}x^2, \quad [ABE] = \frac{1}{2} \cdot 1 \cdot (1 - x).$$

Therefore

$$\frac{[DEF]}{[ABE]} = \frac{x^2}{1 - x} = \frac{2(1 - x)}{1 - x} = 2.$$

The equation from Step 2 is exactly what makes the ratio simplify — no need to find  $x$ . (If you do:  $x = \sqrt{3} - 1$ , so  $x^2 = 4 - 2\sqrt{3}$  and  $1 - x = 2 - \sqrt{3}$ , whose ratio is again 2.)

**Answer: (D) 2**