AMC 10

PRACTICE EXAMINATION

SET 6

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 25 problems are taken from past AMC 10 contests (2000–2025) and appear in their original wording. Advanced level: each problem sits two or three places earlier than it did in its original contest, and problems 21–25 come from original positions 23–25. Problems © Mathematical Association of America.

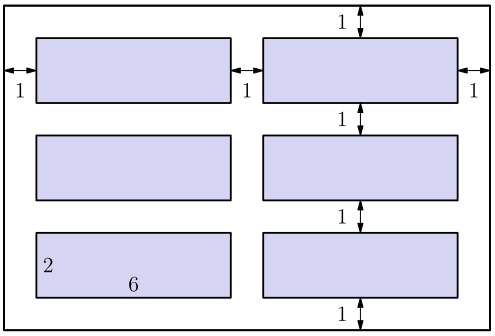
Answer Key and Sources

No.	Ans.	Source	Topic
1	**B**	2017 AMC 10A #3	Algebra
2	**D**	2017 AMC 10B #5	Algebra
3	**E**	2021 Fall AMC 10A #5	Number Theory
4	**D**	2007 AMC 10B #6	Algebra
5	**A**	2020 AMC 10B #7	Number Theory
6	**B**	2009 AMC 10A #9	Algebra
7	**B**	2020 AMC 10A #10	Geometry
8	**C**	2025 AMC 10A #10	Geometry
9	**E**	2005 AMC 10B #12	Counting & Probability
10	**C**	2025 AMC 10B #12	Geometry
11	**C**	2012 AMC 10A #13	Algebra
12	**B**	2013 AMC 10B #15	Geometry
13	**E**	2016 AMC 10B #16	Algebra
14	**B**	2009 AMC 10B #16	Geometry
15	**B**	2013 AMC 10A #18	Geometry
16	**E**	2023 AMC 10B #18	Number Theory
17	**A**	2024 AMC 10B #20	Counting & Probability
18	**D**	2015 AMC 10B #21	Number Theory
19	**B**	2012 AMC 10B #22	Counting & Probability
20	**C**	2018 AMC 10B #22	Counting & Probability
21	**C**	2014 AMC 10A #23	Geometry
22	**B**	2014 AMC 10B #24	Counting & Probability
23	**B**	2022 AMC 10B #24	Algebra
24	**B**	2010 AMC 10A #25	Number Theory
25	**A**	2015 AMC 10A #25	Counting & Probability

1 Algebra

Source: 2017 AMC 10A #3

Tamara has three rows of two 6-feet by 2-feet flower beds in her garden. The beds are separated and also surrounded by 1-foot-wide walkways, as shown on the diagram. What is the total area of the walkways, in square feet?



(A) 72 **(B)** 78 **(C)** 90 **(D)** 120 **(E)** 150

Key idea. The walkways have an awkward shape, but the whole garden is a simple rectangle. Find the area of the whole garden and subtract the area of the flower beds.

Step 1. Across the garden there are 2 beds of length 6 and 3 walkways of width 1 (left, middle, right), so the width is $2 \cdot 6 + 3 \cdot 1 = 15$ feet.

Step 2. Down the garden there are 3 beds of height 2 and 4 walkways of width 1 (top, two in between, bottom), so the height is $3 \cdot 2 + 4 \cdot 1 = 10$ feet.

Step 3. The garden has area $15 \cdot 10 = 150$ square feet. The six beds cover $6 \cdot (6 \cdot 2) = 72$ square feet. The walkways cover the rest: $150 - 72 = 78$ square feet.

Watch out. Both numbers you compute along the way are choices: 150 (E) is the whole garden and 72 (A) is the beds alone. The question asks for what is left over.

Answer: (B) 78

2 Algebra

Source: 2017 AMC 10B #5

> Camilla had twice as many blueberry jelly beans as cherry jelly beans. After eating 10 pieces of each kind, she now has three times as many blueberry jelly beans as cherry jelly beans. How many blueberry jelly beans did she originally have?
>
> **(A)** 10 **(B)** 20 **(C)** 30 **(D)** 40 **(E)** 50

Key idea. Give the unknown counts letters, turn each sentence into an equation, and solve the pair.

Let c be the number of cherry jelly beans and b the number of blueberry jelly beans at the start.

"Twice as many" gives $b = 2c$. After eating 10 of each she has $b - 10$ blueberry and $c - 10$ cherry, and "three times as many" gives $b - 10 = 3(c - 10)$.

Substitute $b = 2c$ into the second equation: $2c - 10 = 3c - 30$, so $c = 20$. Then $b = 2 \cdot 20 = 40$.

Check: she starts with 40 and 20; after eating she has 30 and 10, and $30 = 3 \cdot 10$.

Watch out. The question asks for the *blueberry* beans. Stopping at $c = 20$ gives choice (B).

Answer: (D) 40

3 Number Theory

Source: 2021 Fall AMC 10A #5

> The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit A. What is A?
>
> **(A)** 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Key idea. The problem promises that exactly one choice works, so we do not have to prove $20210A$ is prime — we only have to show the other four choices give composite numbers. Quick divisibility tests do that.

$A = 1$ **or** $A = 7$**.** The digit sum is $2 + 0 + 2 + 1 + 0 + A = 5 + A$, which is 6 or 12. Both are multiples of 3, so 202101 and 202107 are divisible by 3.

$A = 5$**.** The number ends in 5, so 202105 is divisible by 5.

$A = 3$**.** Use the test for 11: alternately add and subtract the digits. For 202103, $2-0+2-1+0-3 = 0$, a multiple of 11. Indeed $202103 = 11 \cdot 18373$.

The only choice left is $A = 9$. (As a check, 202109 really is prime: it has no prime factor up to 449, and $450^2 = 202500 > 202109$.)

Answer: (E) 9

4 Algebra

Source: 2007 AMC 10B #6

> The 2007 AMC 10 will be scored by awarding 6 points for each correct response, 0 points for each incorrect response, and 1.5 points for each problem left unanswered. After looking over the 25 problems, Sarah has decided to attempt the first 22 and leave only the last 3 unanswered. How many of the first 22 problems must she solve correctly in order to score at least 100 points?
>
> **(A)** 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

Key idea. Bank the points that are already guaranteed (the blanks), then ask how many 6-point answers cover the rest.

Step 1. The 3 problems left blank are worth $3 \cdot 1.5 = 4.5$ points no matter what. Wrong answers among the first 22 score 0, so only correct answers add anything.

Step 2. Sarah still needs $100 - 4.5 = 95.5$ points, and each correct answer gives 6. With k correct answers she needs $6k \geq 95.5$, that is, $k \geq \dfrac{95.5}{6} \approx 15.9$.

Step 3. Check the two neighbours. With 15 correct she scores $6 \cdot 15 + 4.5 = 94.5$, which is too few. With 16 correct she scores $6 \cdot 16 + 4.5 = 100.5 \geq 100$. So she needs at least 16.

Watch out. Forgetting the 4.5 points for the blanks means solving $6k \geq 100$, so $k \geq \frac{100}{6} \approx 16.7$ and $k = 17$ — choice (E).

Answer: (D) 16

5 Number Theory

Source: 2020 AMC 10B #7

> How many positive even multiples of 3 less than 2020 are perfect squares?
>
> **(A)** 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 12

Key idea. "Even multiple of 3" just means "multiple of 6". For a perfect square to be divisible by 2 and by 3, its square root must already be divisible by 2 and by 3.

Step 1. If m^2 is divisible by the prime 2, then m is divisible by 2; the same holds for the prime 3. So m is a multiple of 6, say $m = 6k$, and the squares we want are exactly $(6k)^2 = 36k^2$. Conversely, every such number is an even multiple of 3.

Step 2. We need $36k^2 < 2020$, that is, $k^2 < \frac{2020}{36} \approx 56.1$. Since $7^2 = 49 < 56.1 < 64 = 8^2$, the allowed values are $k = 1, 2, \ldots, 7$.

Check. The largest one is $42^2 = 1764 < 2020$, while the next, $48^2 = 2304$, is too big.

So there are 7 such squares: $36, 144, 324, 576, 900, 1296, 1764$.

Answer: (A) 7

---

6 Algebra

Source: 2009 AMC 10A #9

> Positive integers a, b, and 2009, with $a < b < 2009$, form a geometric sequence with an integer ratio. What is a?
>
> **(A)** 7 **(B)** 41 **(C)** 49 **(D)** 287 **(E)** 2009

Key idea. Write the three terms as a, ar, ar^2. Then $ar^2 = 2009$, so r^2 must divide 2009 — factor 2009.

Let the integer ratio be r. The terms are a, $b = ar$, and $ar^2 = 2009$. Because $a < b$, the ratio satisfies $r > 1$.

Factor: $2009 = 7 \times 287 = 7 \times 7 \times 41 = 7^2 \cdot 41$, and 41 is prime. Since r^2 divides $7^2 \cdot 41$ and 41 appears only once, the only perfect square greater than 1 dividing 2009 is $7^2 = 49$. Hence $r = 7$.

Then $a = \dfrac{2009}{49} = 41$ and $b = 41 \cdot 7 = 287$, giving the sequence 41, 287, 2009, which does satisfy $a < b < 2009$.

Watch out. The question asks for a, the *first* term. Reporting the middle term $b = 287$ gives choice (D).

Answer: (B) 41

---

7 Geometry

Source: 2020 AMC 10A #10

> Seven cubes, whose volumes are $1, 8, 27, 64, 125, 216$, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?
>
> **(A)** 644 **(B)** 658 **(C)** 664 **(D)** 720 **(E)** 749

Key idea. Add up the surface areas of the seven separate cubes, then take away exactly the area that gets hidden where two cubes touch.

Step 1. The volumes are $1^3, 2^3, \ldots, 7^3$, so the edges are $1, 2, \ldots, 7$. Separately the cubes have total surface area

$$6\left(1^2 + 2^2 + \cdots + 7^2\right) = 6 \cdot 140 = 840.$$

Step 2. Where a cube of edge k sits on the next larger cube, two squares of area k^2 disappear: the bottom face of the small cube, and the equal patch of the large cube's top that it covers. The contacts involve $k = 1, 2, \ldots, 6$, so the hidden area is

$$2\left(1^2 + 2^2 + \cdots + 6^2\right) = 2 \cdot 91 = 182.$$

Step 3. The surface area of the tower is $840 - 182 = 658$.

Another way. Count what you see. The four vertical sides of every cube are fully exposed: $4 \cdot 140 = 560$. Looking straight down, the exposed tops together fill a 7×7 square, area 49, and the bottom adds another 49. Total: $560 + 49 + 49 = 658$.

Watch out. Subtracting each contact area only once gives $840 - 91 = 749$, choice (E). Every contact hides a face on *both* cubes.

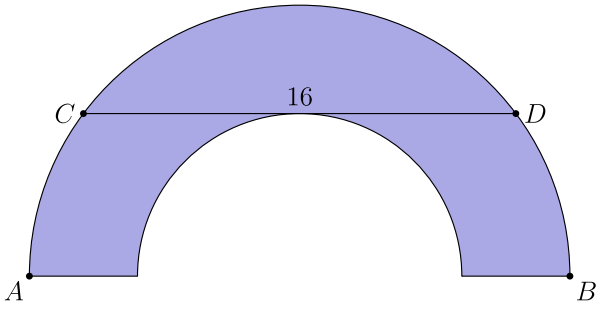
Answer: (B) 658

---

8 Geometry

Source: 2025 AMC 10A #10

A semicircle has diameter AB and chord CD of length 16 parallel to AB. A smaller semicircle with diameter on AB and tangent to CD is cut from the larger semicircle, as shown below.



What is the area of the resulting figure, shown shaded?

(A) 16π **(B)** 24π **(C)** 32π **(D)** 48π **(E)** 64π

Key idea. We never need the two radii separately — only the difference of their squares. A right triangle built from the big radius, half the chord, and the small radius gives exactly that.

Step 1. Let O be the center of the large semicircle (the midpoint of AB), R its radius, and r the radius of the small semicircle. Let P be the midpoint of CD, so $PD = 8$ and $OP \perp CD$ (the perpendicular from a center bisects a chord).

Step 2. The small semicircle has its center on AB and touches the line CD, which is parallel to AB. So its radius is the distance between the two parallel lines: $r = OP$.

Step 3. In right triangle OPD, $OD = R$, so

$$R^2 = OP^2 + PD^2 = r^2 + 64, \qquad \text{i.e.} \qquad R^2 - r^2 = 64.$$

Step 4. The shaded area is the large semicircle minus the small one:

$$\tfrac{1}{2}\pi R^2 - \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi\left(R^2 - r^2\right) = \tfrac{1}{2}\pi \cdot 64 = 32\pi.$$

Watch out. These are *semi*circles. Forgetting the factor $\frac{1}{2}$ gives $\pi(R^2 - r^2) = 64\pi$, choice (E).

Answer: (C) 32π

---

9 Counting & Probability

Source: 2005 AMC 10B #12

Twelve fair dice are rolled. What is the probability that the product of the numbers on the top faces is prime?

(A) $\left(\frac{1}{12}\right)^{12}$ **(B)** $\left(\frac{1}{6}\right)^{12}$ **(C)** $2\left(\frac{1}{6}\right)^{11}$ **(D)** $\frac{5}{2}\left(\frac{1}{6}\right)^{11}$ **(E)** $\left(\frac{1}{6}\right)^{10}$

Key idea. A product of whole numbers is prime only if one factor is a prime and all the others are 1. So exactly one die shows 2, 3 or 5, and the other eleven show 1.

Step 1. Choose which die shows the prime: 12 ways.

Step 2. That die shows 2, 3 or 5 with probability $\frac{3}{6} = \frac{1}{2}$. Each of the other eleven dice shows 1 with probability $\frac{1}{6}$.

Step 3. Multiply:

$$12 \cdot \frac{1}{2} \cdot \left(\frac{1}{6}\right)^{11} = 6 \cdot \left(\frac{1}{6}\right)^{11} = \left(\frac{1}{6}\right)^{10}.$$

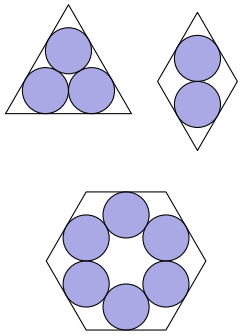
Watch out. Counting only one prime face (probability $\frac{1}{6}$ instead of $\frac{3}{6}$) gives $12 \cdot \frac{1}{6} \cdot \left(\frac{1}{6}\right)^{11} = 2\left(\frac{1}{6}\right)^{11}$, choice (C). Remember there are three prime faces: 2, 3 and 5.

Answer: (E) $\left(\frac{1}{6}\right)^{10}$

10 Geometry

Source: 2025 AMC 10B #12

The figure below shows an equilateral triangle, a rhombus with a 60° angle, and a regular hexagon, each of them containing some mutually tangent congruent disks. Let T, R, and H, respectively, denote the ratio in each case of the total area of the disks to the area of the enclosing polygon.



Which of the following is true?

(A) $T = H = R$ **(B)** $H < R = T$ **(C)** $H = R < T$ **(D)** $H < R < T$ **(E)** $H < T < R$

Key idea. Ratios do not depend on size, so in each picture give the disks radius $r = 1$, find the polygon's side from the tangencies, and compare the three numbers. Every tangency to a side of a 60° corner turns into a 30-60-90 triangle.

Step 1: triangle. A disk tangent to the base and a side sits in a 60° corner; its center is 1 above the base and $\frac{1}{\tan 30^\circ} = \sqrt{3}$ in from the corner. The two bottom disks touch, so their centers are 2 apart. Hence the side is $s = \sqrt{3} + 2 + \sqrt{3} = 2 + 2\sqrt{3}$. (The top disk, centered on the axis 2 below the apex, is at distance 2 from each bottom center, so it fits.) Then

$$T = \frac{3\pi}{\frac{\sqrt{3}}{4}(2+2\sqrt{3})^2} = \frac{3\pi}{\sqrt{3}\,(4+2\sqrt{3})} = \frac{(2\sqrt{3}-3)\pi}{2} \approx 0.729.$$

Step 2: rhombus. The two disks lie along the long diagonal, touching each other at the rhombus's center, each also touching the two sides at a 60° vertex. A center at distance d from that vertex is $d\sin 30^\circ = \frac{d}{2}$ from the sides, so $d = 2$; the half-diagonal is $d + 1 = 3$, and the long diagonal is 6. For side a the long diagonal is $a\sqrt{3}$, so $a = 2\sqrt{3}$. The area is $a^2 \sin 60^\circ = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$, and

$$R = \frac{2\pi}{6\sqrt{3}} = \frac{\pi\sqrt{3}}{9} \approx 0.605.$$

Step 3: hexagon. Each disk touches a side at its midpoint, so its center lies on an apothem at distance c from the hexagon's center. Neighboring centers are 60° apart, forming an equilateral triangle with the hexagon's center, so $c = 2$. The apothem is $c + 1 = 3$, so for side a, $\frac{\sqrt{3}}{2}a = 3$, $a = 2\sqrt{3}$, and the area is $\frac{3\sqrt{3}}{2}a^2 = 18\sqrt{3}$. Then

$$H = \frac{6\pi}{18\sqrt{3}} = \frac{\pi\sqrt{3}}{9} \approx 0.605.$$

So $H = R < T$ (indeed $0.605 < 0.729$).

Answer: (C) $H = R < T$

---

11 Algebra

Source: 2012 AMC 10A #13

An *iterative average* of the numbers $1, 2, 3, 4$, and 5 is computed the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

(A) $\frac{31}{16}$ **(B)** 2 **(C)** $\frac{17}{8}$ **(D)** 3 **(E)** $\frac{65}{16}$

Key idea. Write the whole procedure as one formula. Each new averaging halves the weight of everything before it, so later numbers count more — put big numbers last to maximize and first to minimize.

Step 1. Call the order a, b, c, d, e. The successive means are

$$\frac{a+b}{2}, \qquad \frac{a+b+2c}{4}, \qquad \frac{a+b+2c+4d}{8}, \qquad \frac{a+b+2c+4d+8e}{16}.$$

So the final value is $\dfrac{a+b+2c+4d+8e}{16}$: the weights are $1, 1, 2, 4, 8$.

Step 2. To make the value as large as possible, give the largest weight to the largest number: $e = 5$, $d = 4$, $c = 3$, and $\{a, b\} = \{1, 2\}$. This gives

$$\frac{1+2+2\cdot 3+4\cdot 4+8\cdot 5}{16} = \frac{1+2+6+16+40}{16} = \frac{65}{16}.$$

Step 3. To make it as small as possible, reverse: $e = 1$, $d = 2$, $c = 3$, $\{a, b\} = \{4, 5\}$:

$$\frac{5+4+2\cdot 3+4\cdot 2+8\cdot 1}{16} = \frac{5+4+6+8+8}{16} = \frac{31}{16}.$$

(Pairing the largest weights with the largest numbers is best because swapping any out-of-order pair can only raise the total; checking all 120 orders confirms these extremes.)

Step 4. The difference is $\dfrac{65}{16} - \dfrac{31}{16} = \dfrac{34}{16} = \dfrac{17}{8}$.

Watch out. The two extremes themselves, $\frac{31}{16}$ and $\frac{65}{16}$, are choices (A) and (E). The question asks for their *difference*.

Answer: (C) $\frac{17}{8}$

12 Geometry

Source: 2013 AMC 10B #15

A wire is cut into two pieces, one of length a and the other of length b. The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$?

(A) 1 **(B)** $\frac{\sqrt{6}}{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $\frac{3\sqrt{2}}{2}$

Key idea. A regular hexagon is six equilateral triangles. So both areas can be written with the single formula $\frac{\sqrt{3}}{4}s^2$ for an equilateral triangle of side s, and setting them equal gives a ratio of squares.

Step 1. The triangle has perimeter a, so its side is $\frac{a}{3}$ and its area is

$$\frac{\sqrt{3}}{4}\left(\frac{a}{3}\right)^2 = \frac{\sqrt{3}}{4}\cdot\frac{a^2}{9}.$$

Step 2. The hexagon has perimeter b, so its side is $\frac{b}{6}$. It splits into 6 equilateral triangles of that side, so its area is

$$6\cdot\frac{\sqrt{3}}{4}\left(\frac{b}{6}\right)^2 = \frac{\sqrt{3}}{4}\cdot\frac{6b^2}{36} = \frac{\sqrt{3}}{4}\cdot\frac{b^2}{6}.$$

Step 3. Setting the areas equal and cancelling $\frac{\sqrt{3}}{4}$:

$$\frac{a^2}{9} = \frac{b^2}{6} \quad\Longrightarrow\quad \frac{a^2}{b^2} = \frac{9}{6} = \frac{3}{2}.$$

Step 4. Taking the positive square root,

$$\frac{a}{b} = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2} \approx 1.22.$$

Watch out. Areas scale with the *square* of lengths. Stopping at $\frac{a^2}{b^2} = \frac{3}{2}$ and reporting $\frac{3}{2}$ is wrong — and $\frac{3}{2}$ is not among the choices, a sign the square root is missing.

Another way. Think in terms of scaling. Let the hexagon have side s, so $b = 6s$ and its area is 6 times that of an equilateral triangle of side s. A triangle with 6 times the area has $\sqrt{6}$ times the side, so the triangle's side is $\sqrt{6}\,s$ and $a = 3\sqrt{6}\,s$. Then $\frac{a}{b} = \frac{3\sqrt{6}\,s}{6s} = \frac{\sqrt{6}}{2}$.

Answer: (B) $\frac{\sqrt{6}}{2}$

13 Algebra

Source: 2016 AMC 10B #16

The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

(A) $\frac{1+\sqrt{5}}{2}$ **(B)** 2 **(C)** $\sqrt{5}$ **(D)** 3 **(E)** 4

Key idea. Write S in terms of the common ratio r alone; the condition "second term $= 1$" lets you eliminate the first term. Then minimizing S becomes maximizing a quadratic, which completing the square handles.

Step 1. Let the first term be a and the ratio r, with $|r| < 1$ so the series converges. The second term is $ar = 1$, so $r \neq 0$ and $a = \frac{1}{r}$. Then

$$S = \frac{a}{1-r} = \frac{1}{r(1-r)}.$$

Step 2. S must be positive, so $r(1-r) > 0$, which (with $|r| < 1$) means $0 < r < 1$. For a positive denominator, S is smallest when $r(1-r)$ is largest.

Step 3. Complete the square:

$$r(1-r) = r - r^2 = \tfrac{1}{4} - \left(r - \tfrac{1}{2}\right)^2 \leq \tfrac{1}{4},$$

with equality exactly when $r = \frac{1}{2}$. Hence $S \geq \frac{1}{1/4} = 4$.

Step 4: It is achieved. With $r = \frac{1}{2}$ we get $a = 2$, and the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ has second term 1 and sum $\frac{2}{1-\frac{1}{2}} = 4$.

Watch out. Do not forget the word *positive*. With $-1 < r < 0$ the sum $\frac{1}{r(1-r)}$ is negative and can be made as low as you like, so the restriction $S > 0$ is what makes a smallest value exist.

Answer: (E) 4

14 Geometry

Source: 2009 AMC 10B #16

Points A and C lie on a circle centered at O, each of $\overline{BA}$ and $\overline{BC}$ are tangent to the circle, and $\triangle ABC$ is equilateral. The circle intersects $\overline{BO}$ at D. What is $\frac{BD}{BO}$?

(A) $\frac{\sqrt{2}}{3}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{3}}{3}$ **(D)** $\frac{\sqrt{2}}{2}$ **(E)** $\frac{\sqrt{3}}{2}$

Key idea. A tangent line is perpendicular to the radius at the point of contact. That gives right triangles at A and C, and the 60° angle at B makes them 30-60-90 triangles.

Step 1. Let the radius be r, so $OA = OC = r$. Because $\overline{BC}$ is tangent at C, $\angle OCB = 90^\circ$; likewise $\angle OAB = 90^\circ$.

Step 2. Right triangles BAO and BCO share the hypotenuse $\overline{BO}$ and have equal legs $OA = OC$, so they are congruent. Hence $\overline{BO}$ splits $\angle ABC = 60^\circ$ into two equal parts: $\angle OBC = 30^\circ$.

Step 3. In right triangle BCO the side opposite the 30° angle is $OC = r$, and in a 30-60-90 triangle the hypotenuse is twice that side. So $BO = 2r$.

Step 4. D is on the circle, so $OD = r$, and D lies between B and O since $BO = 2r > r$. Thus $BD = BO - OD = 2r - r = r$, and

$$\frac{BD}{BO} = \frac{r}{2r} = \frac{1}{2}.$$

In other words, D is the midpoint of $\overline{BO}$.

Answer: (B) $\frac{1}{2}$

15 Geometry

Source: 2013 AMC 10A #18

Let points $A = (0,0), B = (1,2), C = (3,3)$, and $D = (4,0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects $\overline{CD}$ at point $\left(\frac{p}{q}, \frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p + q + r + s$?

(A) 54 **(B)** 58 **(C)** 62 **(D)** 70 **(E)** 75

Key idea. The cutting line from A to a point G on $\overline{CD}$ chops off triangle AGD. That triangle has base AD on the x-axis, so its area depends only on the *height* of G — and it must be half the area of $ABCD$.

Step 1. Find the area of $ABCD$. Drop perpendiculars from B and C to the x-axis, at $(1,0)$ and $(3,0)$. This splits $ABCD$ into a triangle, a trapezoid, and a triangle:

$$\tfrac{1}{2} \cdot 1 \cdot 2 + \tfrac{1}{2}(2+3) \cdot 2 + \tfrac{1}{2} \cdot 1 \cdot 3 = 1 + 5 + \tfrac{3}{2} = \tfrac{15}{2}.$$

Step 2. Let the line meet $\overline{CD}$ at G, with height y_G. Triangle AGD has base $AD = 4$ and must have area $\frac{1}{2} \cdot \frac{15}{2} = \frac{15}{4}$:

$$\tfrac{1}{2} \cdot 4 \cdot y_G = \tfrac{15}{4} \quad \Longrightarrow \quad y_G = \tfrac{15}{8}.$$

The other piece, $ABCG$, then has the remaining $\frac{15}{4}$.

Step 3. Line CD passes through $(3, 3)$ and $(4, 0)$, so its slope is -3 and its equation is $y = -3x + 12$. Setting $y = \frac{15}{8}$:

$$3x = 12 - \tfrac{15}{8} = \tfrac{81}{8}, \qquad x = \tfrac{27}{8}.$$

This lies between 3 and 4, so $G = \left(\frac{27}{8}, \frac{15}{8}\right)$ really is on segment $\overline{CD}$.

Step 4. Both fractions are in lowest terms, so $p + q + r + s = 27 + 8 + 15 + 8 = 58$.

Answer: (B) 58

---

16 Number Theory

Source: 2023 AMC 10B #18

Suppose a, b, and c are positive integers such that

$$\frac{a}{14} + \frac{b}{15} = \frac{c}{210}.$$

Which of the following statements are necessarily true?

I. If $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both, then $\gcd(c, 210) = 1$.

II. If $\gcd(c, 210) = 1$, then $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both.

III. $\gcd(c, 210) = 1$ if and only if $\gcd(a, 14) = \gcd(b, 15) = 1$.

(A) I, II, and III **(B)** I only **(C)** I and II only

(D) III only **(E)** II and III only

Key idea. Clear the fractions to get c as a combination of a and b. Then "$\gcd(c, 210) = 1$" just means c avoids the four primes $2, 3, 5, 7$, and each of those primes can be checked separately.

Step 1: clear denominators. Multiply both sides by $210 = 14 \cdot 15$:

$$c = 15a + 14b.$$

Step 2: one prime at a time. Since $210 = 2 \cdot 3 \cdot 5 \cdot 7$, $\gcd(c, 210) = 1$ means none of $2, 3, 5, 7$ divides c.

- Mod 2: $15a \equiv a$ and $14b \equiv 0$, so $2 \mid c \iff 2 \mid a$.
- Mod 7: $15a \equiv a$ and $14b \equiv 0$, so $7 \mid c \iff 7 \mid a$.
- Mod 3: $15a \equiv 0$ and $14b \equiv 2b$, so $3 \mid c \iff 3 \mid b$.
- Mod 5: $15a \equiv 0$ and $14b \equiv 4b$, so $5 \mid c \iff 5 \mid b$.

Step 3: combine. So $\gcd(c, 210) = 1$ exactly when a has no factor 2 or 7 *and* b has no factor 3 or 5, that is, when $\gcd(a, 14) = 1$ and $\gcd(b, 15) = 1$. This is statement III, so III is true.

Step 4: the other two. II: if $\gcd(c, 210) = 1$, then by III both gcds equal 1, so certainly "one or the other or both" holds. II is true.

I is false. Take $a = 1$, $b = 3$: then $\gcd(a, 14) = 1$, but $c = 15 + 42 = 57 = 3 \cdot 19$, so $\gcd(c, 210) = 3$. One coprime condition is not enough; you need both.

Only II and III are necessarily true.

Watch out. Testing I only on friendly numbers such as $a = b = 1$ (so $c = 29$, coprime to 210) makes it look true, and with II and III that leads to (A). A "necessarily true" claim needs a proof; one counterexample kills it.

Answer: (E) II and III only

---

17 Counting & Probability

Source: 2024 AMC 10B #20

> Three different pairs of shoes are placed in a row so that no left shoe is next to a right shoe from a different pair. In how many ways can these six shoes be lined up?
>
> **(A)** 60 **(B)** 72 **(C)** 90 **(D)** 108 **(E)** 120

Key idea. First look only at the pattern of L's (left shoes) and R's (right shoes) along the row. The rule bites exactly where an L meets an R — call that a *switch* — because the two shoes there must be a matching pair. So decide which patterns are possible, then count how many ways to fill each one.

Step 1: what a switch forces. At every switch the two shoes form a pair. Suppose a run of a single letter sits in the middle of the row, like the R in L R L. That shoe has a switch on both sides, so it would have to be the mate of both neighbours — impossible, since it has only one mate. So *an interior run cannot have length* 1 (a run at either end may). Neighbours with the same letter (LL or RR) are never a problem. Also, two switches never share a shoe, and a pair has only two shoes, so different switches use different pairs.

Step 2: the possible patterns. We need three L's and three R's. Starting with L, the runs alternate L, R, L, …:

- 2 runs: LLLRRR.
- 3 runs (L, R, L): the R run is RRR; the L's split $1 + 2$ or $2 + 1$: LRRRLL, LLRRRL.
- 4 runs (L, R, L, R): the two middle runs need length ≥ 2, so they are 2 and 2, leaving 1 and 1 at the ends: LRRLLR.
- 5 or more runs: three L runs from only three L's means a middle L run of length 1 — not allowed.

Swapping L and R gives four more patterns, 8 in all.

Step 3: fill each pattern. With one switch (LLLRRR, RRRLLL): choose the pair at the switch (3 ways; the pattern fixes which of its shoes is on which side), then order the other two left shoes (2) and the other two right shoes (2): 12 ways each. With two switches: assign pairs to the switches in $3 \cdot 2 = 6$ ways; the last pair's left and right shoes each have exactly one empty slot left, so 6 ways. With three switches (LRRLLR, RLLRRL): every shoe is at a switch, and the pairs can be assigned in $3! = 6$ ways.

Step 4. Total: $2 \cdot 12 + 4 \cdot 6 + 2 \cdot 6 = 24 + 24 + 12 = 60$.

Answer: (A) 60

---

18 Number Theory

Source: 2015 AMC 10B #21

> Cozy the Cat and Dash the Dog are going up a staircase with a certain number of steps. However, instead of walking up the steps one at a time, both Cozy and Dash jump.
>
> Cozy goes two steps up with each jump (though if necessary, he will just jump the last step).
>
> Dash goes five steps up with each jump (though if necessary, he will just jump the last steps if there are fewer than 5 steps left).
>
> Suppose Dash takes 19 fewer jumps than Cozy to reach the top of the staircase. Let s denote the sum of all possible numbers of steps this staircase can have. What is the sum of the digits of s?
>
> **(A)** 9 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 15

Key idea. With t steps, Cozy needs $\lceil t/2 \rceil$ jumps and Dash needs $\lceil t/5 \rceil$ (round up for the short last jump). Roughly, the difference is $\frac{t}{2} - \frac{t}{5} = \frac{3t}{10}$, so t is near $\frac{190}{3} \approx 63$. Bound the rounding error to get a short list, then check each value.

Step 1: narrow down t. Rounding up adds less than 1, so $\frac{t}{2} \le \lceil \frac{t}{2} \rceil \le \frac{t+1}{2}$ and $\frac{t}{5} \le \lceil \frac{t}{5} \rceil \le \frac{t+4}{5}$. Hence the difference $D = \lceil \frac{t}{2} \rceil - \lceil \frac{t}{5} \rceil$ satisfies

$$\frac{t}{2} - \frac{t+4}{5} \le D \le \frac{t+1}{2} - \frac{t}{5}, \quad \text{i.e.} \quad \frac{3t-8}{10} \le D \le \frac{3t+5}{10}.$$

Setting $D = 19$: $3t - 8 \le 190$ gives $t \le 66$, and $3t + 5 \ge 190$ gives $t \ge 61\frac{2}{3}$. So $t \in \{62, 63, 64, 65, 66\}$.

Step 2: check each.

t	62	63	64	65	66
Cozy $\lceil t/2 \rceil$	31	32	32	33	33
Dash $\lceil t/5 \rceil$	13	13	13	13	14
difference	18	19	19	20	19

The possible staircases have 63, 64 and 66 steps.

Step 3. $s = 63 + 64 + 66 = 193$, and the sum of its digits is $1 + 9 + 3 = 13$.

Watch out. Stopping after finding some of the solutions. Missing $t = 64$, for instance, gives $s = 129$ and digit sum 12 — choice (C). The difference does not grow steadily, so every candidate must be checked.

Another way. Let Dash take $d + 1$ jumps, so $t \in \{5d + 1, \ldots, 5d + 5\}$. Cozy takes $d + 20$ jumps, so $t = 2d + 39$ or $2d + 40$. The equations with whole-number solutions are $5d + 3 = 2d + 39$ ($d = 12$, $t = 63$), $5d + 4 = 2d + 40$ ($d = 12$, $t = 64$) and $5d + 1 = 2d + 40$ ($d = 13$, $t = 66$).

Answer: (D) 13

---

19 Counting & Probability

Source: 2012 AMC 10B #22

Let $(a_1, a_2, \ldots, a_{10})$ be a list of the first 10 positive integers such that for each $2 \le i \le 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. How many such lists are there?

(A) 120 **(B)** 512 **(C)** 1024 **(D)** 181,440 **(E)** 362,880

Key idea. Write out a few valid lists and look at the numbers used so far: they always form a block of consecutive integers with no gaps. Each new number must be a neighbour of the block, so it extends the block one step to the left or one step to the right. Counting lists becomes counting left/right choices.

Step 1 (the used numbers form a block). The first term alone is a block. Suppose $a_1, \ldots, a_{i-1}$ are exactly the integers $\ell, \ell+1, \ldots, h$. The new term a_i is not among them, yet a_i+1 or a_i-1 is. The only numbers outside the block with a neighbour inside it are $\ell - 1$ and $h + 1$, so a_i is one of these and the block grows by one at an end. By induction, every prefix is a block.

Step 2 (what a list is). So a valid list is: a starting number a_1, then 9 moves, each "extend left" (write the number just below the block) or "extend right" (write the number just above). Conversely, any such run of moves obeys the rule, because each new number is next to an end of the block. The list must end with the block $1, 2, \ldots, 10$.

Step 3 (count). If $a_1 = k$, the block must grow left $k - 1$ times (to reach 1) and right $10 - k$ times (to reach 10). Choosing which of the 9 moves go left gives $\binom{9}{k-1}$ lists. Summing over $k = 1, \ldots, 10$,

$$\binom{9}{0} + \binom{9}{1} + \cdots + \binom{9}{9} = 2^9 = 512.$$

For instance, $a_1 = 1$ forces the single list $1, 2, \ldots, 10$, matching $\binom{9}{0} = 1$.

Another way. Read the list backwards. The last term a_{10} must be 1 or 10: an interior value v appearing last would leave both $v - 1$ and $v + 1$ already used, so the numbers before it could not have been a block. Removing it leaves a block again, so each of $a_{10}, a_9, \ldots, a_2$ is the smallest or largest remaining number — 2 choices each — and a_1 is then forced. That is $2^9 = 512$.

Watch out. Allowing two choices for every one of the 10 positions, including the first, gives $2^{10} = 1024$ — choice (C). Once $a_2, \ldots, a_{10}$ are chosen, a_1 is the one number left, so it is not a free choice.

Answer: (B) 512

---

20 Counting & Probability

Source: 2018 AMC 10B #22

> Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. Which of the following numbers is closest to the probability that x, y, and 1 are the side lengths of an obtuse triangle?
>
> **(A)** 0.21 **(B)** 0.25 **(C)** 0.29 **(D)** 0.50 **(E)** 0.79

Key idea. Two random numbers in $[0, 1]$ are a random point in the unit square, so the probability is an *area*. Turn "triangle" and "obtuse" into inequalities in x and y and shade the region.

Step 1. Triangle. Since $x, y \leq 1$, the side 1 is the longest. The inequalities $x + 1 > y$ and $y + 1 > x$ hold automatically (except on a set of zero area), so the only real condition is $x + y > 1$: the point lies above the line from $(1, 0)$ to $(0, 1)$.

Step 2. Obtuse. An obtuse angle must sit opposite the longest side, here the side 1. By the law of cosines, that angle is obtuse exactly when $x^2 + y^2 < 1^2$: the point lies inside the quarter circle of radius 1 centered at the origin.

Step 3. Area. The region inside the quarter disk but above the line $x + y = 1$ is the quarter disk (area $\frac{\pi}{4}$) minus the right triangle with vertices $(0, 0), (1, 0), (0, 1)$ (area $\frac{1}{2}$), which lies entirely inside the quarter disk. So the probability is

$$\frac{\pi}{4} - \frac{1}{2} \approx 0.7854 - 0.5 = 0.2854.$$

The closest choice is 0.29.

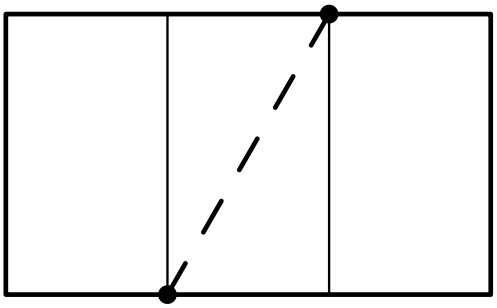
Watch out. Forgetting the triangle condition gives just the quarter disk, $\frac{\pi}{4} \approx 0.79$ (E). Shading the *acute* triangles instead (outside the quarter circle) gives $1 - \frac{\pi}{4} \approx 0.21$ (A).

Answer: (C) 0.29

21 Geometry

Source: 2014 AMC 10A #23

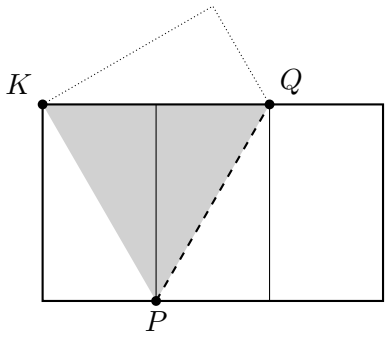
A rectangular piece of paper whose length is $\sqrt{3}$ times the width has area A. The paper is divided into three equal sections along the opposite lengths, and then a dotted line is drawn from the first divider to the second divider on the opposite side as shown. The paper is then folded flat along this dotted line to create a new shape with area B. What is the ratio $B : A$?



(A) $1 : 2$ **(B)** $3 : 5$ **(C)** $2 : 3$ **(D)** $3 : 4$ **(E)** $4 : 5$

Key idea. Folding never changes the paper's area; it only stacks some paper on top of other paper. So $B = A -$ (overlap), and the whole problem is to find the overlap. The $\sqrt{3}$ in the statement hints that 60° angles and equilateral triangles are coming.

Step 1: coordinates. The ratio does not depend on size, so take width 1 and length $\sqrt{3}$; then $A = \sqrt{3}$. Put the rectangle with corners $(0,0)$ and $(\sqrt{3},1)$. The dividers are at $x = \frac{\sqrt{3}}{3}$ and $x = \frac{2\sqrt{3}}{3}$, so the fold line runs from $P = \left(\frac{\sqrt{3}}{3}, 0\right)$ to $Q = \left(\frac{2\sqrt{3}}{3}, 1\right)$. Let $K = (0,1)$ be the top-left corner. We fold the right-hand part over onto the left.



Step 2: the fold angle. The fold rises 1 over a run of $\frac{\sqrt{3}}{3}$, so its slope is $\sqrt{3}$ and it makes a 60° angle with the bottom edge. Its length is $PQ = \sqrt{\frac{1}{3} + 1} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

Step 3: where the bottom edge lands. The bottom edge to the right of P has length $\sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$ and makes 60° with PQ. Folding reflects it to the other side of PQ, still at 60°, so it now points from P at $60^\circ + 60^\circ = 120^\circ$ from the bottom edge — straight toward K. Since $PK = \sqrt{\frac{1}{3} + 1} = \frac{2\sqrt{3}}{3}$ as well, the bottom-right corner lands exactly on K.

Step 4: the overlap. The folded flap is a quadrilateral with vertices P, K, Q and one more vertex above the paper (dotted). The segment KQ lies on the top edge, so the part of the flap lying on the paper is exactly $\triangle PKQ$ (shaded). Its sides are $PK = PQ = KQ = \frac{2\sqrt{3}}{3}$, so it is equilateral with area

$$\frac{\sqrt{3}}{4}\left(\frac{2\sqrt{3}}{3}\right)^2 = \frac{\sqrt{3}}{4}\cdot\frac{4}{3} = \frac{\sqrt{3}}{3}.$$

Step 5. The folded shape covers $B = A - \frac{\sqrt{3}}{3} = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$, so

$$B : A = \frac{2\sqrt{3}}{3} : \sqrt{3} = 2 : 3.$$

Answer: (C) $2 : 3$

---

22 Counting & Probability

Source: 2014 AMC 10B #24

> The numbers $1, 2, 3, 4, 5$ are to be arranged in a circle. An arrangement is *bad* if it is not true that for every n from 1 to 15 one can find a subset of the numbers that appear consecutively on the circle that sum to n. Arrangements that differ only by a rotation or a reflection are considered the same. How many different bad arrangements are there?
>
> **(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5.

Key idea. Use complements. On a circle, the numbers *not* in a block of consecutive numbers also form a consecutive block, and the two sums add to 15. So sum n is reachable exactly when $15 - n$ is, and almost all of the 15 targets come for free — only 6 and 7 need real checking.

Step 1. Reduce to 6 and 7. The single numbers give 1 to 5, and all five give 15. By complements, 10 to 14 are reachable as well, and $8, 9$ are reachable exactly when $7, 6$ are. So an arrangement is bad exactly when it has *no* block summing to 6, or *no* block summing to 7.

Step 2. No block sums to 6. The ways to make 6 from distinct numbers in $\{1, \ldots, 5\}$ are $1 + 5$, $2 + 4$, $1 + 2 + 3$. A block $\{1, 2, 3\}$ exists exactly when its complement $\{4, 5\}$ is a block, i.e. 4 and 5 are neighbors. So the conditions are: 5 is not next to 1; 2 is not next to 4; 5 is not next to 4.

Then both neighbors of 5 come from $\{2, 3\}$, giving $2, 5, 3$ in a row. The neighbors of 4 cannot be 2 or 5, so they are 3 and 1: the circle is $2, 5, 3, 4, 1$. It works — no block of it sums to 6 — and it is the only one.

Step 3. No block sums to 7. Now $7 = 2 + 5 = 3 + 4 = 1 + 2 + 4$, and $\{1, 2, 4\}$ is a block exactly when 3 and 5 are neighbors. So 5 is next to neither 2 nor 3, and 3 is not next to 4. The neighbors of 5 are 1 and 4; the neighbors of 3 come from $\{1, 2\}$, giving $4, 5, 1, 3, 2$. Again exactly one circle, and it has no block summing to 7.

Step 4. The two circles are different (in the first 5 sits between 2 and 3, in the second between 1 and 4), so there are exactly 2 bad arrangements. A computer check of all 12 arrangements agrees.

Watch out. Forgetting that reflections count as the same doubles each bad circle (read clockwise vs. counterclockwise) and gives 4, choice (D).

Answer: (B) 2

---

23 Algebra

Source: 2022 AMC 10B #24

Consider functions f that satisfy

$$|f(x) - f(y)| \le \frac{1}{2}|x - y|$$

for all real numbers x and y. Of all such functions that also satisfy the equation $f(300) = f(900)$, what is the greatest possible value of

$$f(f(800)) - f(f(400))?$$

(A) 25 **(B)** 50 **(C)** 100 **(D)** 150 **(E)** 200

Key idea. The condition says f shrinks every distance by at least half. Applying f twice therefore shrinks distances to a quarter. The given equality $f(300) = f(900)$ gives a common "meeting point", so route the comparison of $f(f(800))$ and $f(f(400))$ through it — and then build an f that attains the bound.

Step 1: shrinking twice. Using the condition with 400 and 300, then again with $f(400)$ and $f(300)$:

$$|f(f(400)) - f(f(300))| \le \tfrac{1}{2}|f(400) - f(300)| \le \tfrac{1}{4}|400 - 300| = 25.$$

In the same way, $|f(f(800)) - f(f(900))| \le \frac{1}{4}|800 - 900| = 25$.

Step 2: the meeting point. Since $f(300) = f(900)$, also $f(f(300)) = f(f(900))$; call this number M. Then

$$f(f(800)) - f(f(400)) = \big(f(f(800)) - M\big) + \big(M - f(f(400))\big) \le 25 + 25 = 50.$$

Step 3: the bound is achieved. Equality needs every inequality above to be tight. One way to arrange this: let $f(400)$ and $f(800)$ sit 50 below and 50 above $f(300)$, and let f climb at the full rate $\frac{1}{2}$ between those two values. One function that does this: take f to be the broken line through

$$(300, 600),\ (400, 550),\ (550, 575),\ (650, 625),\ (800, 650),\ (900, 600),$$

with $f(x) = 600$ for $x \le 300$ and for $x \ge 900$. The slopes of the pieces are $-\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{2}$, $\frac{1}{6}$, $-\frac{1}{2}$ (and 0 outside), all at most $\frac{1}{2}$ in size, so $|f(x) - f(y)| \le \frac{1}{2}|x - y|$ for all x, y. Also $f(300) = f(900) = 600$.

Now $f(400) = 550$ and $f(800) = 650$, so

$$f(f(800)) - f(f(400)) = f(650) - f(550) = 625 - 575 = 50.$$

Both halves are done: the value never exceeds 50, and 50 is reached.

Watch out. Skipping the meeting point and using only the shrinking gives $|f(f(800)) - f(f(400))| \le \frac{1}{4} \cdot 400 = 100$. That bound is true but cannot be reached once $f(300) = f(900)$; picking 100, choice (C), is the usual mistake.

Answer: (B) 50

---

24 Number Theory

Source: 2010 AMC 10A #25

Jim starts with a positive integer n and creates a sequence of numbers. Each successive number is obtained by subtracting the largest possible integer square less than or equal to the current number until zero is reached. For example, if Jim starts with $n = 55$, then his sequence contains 5 numbers:

$$\begin{array}{rcrcr} & & & & 55 \\ 55 & - & 7^2 & = & 6 \\ 6 & - & 2^2 & = & 2 \\ 2 & - & 1^2 & = & 1 \\ 1 & - & 1^2 & = & 0 \end{array}$$

Let N be the smallest number for which Jim's sequence has 8 numbers. What is the units digit of N?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Key idea. The sequence that starts at n is just n followed by the sequence that starts at the next number. So the smallest start with 8 numbers is built from the smallest start with 7 numbers, and so on: work *backwards* from 0. The tool that controls each backward step is how big a remainder can be after removing the largest square.

Step 1 (how big can the next number be?). Let s^2 be the largest square with $s^2 \le n$, so $n < (s+1)^2 = s^2 + 2s + 1$. The next number is $n - s^2$, and it satisfies $n - s^2 \le 2s$.

Step 2 (the recursion). Let m_k be the smallest start whose sequence has k numbers (counting n and the final 0). Starting at 1 gives $1, 0$, so $m_2 = 1$.

Lower bound. If n has $k+1$ numbers, the next number $n' = n - s^2$ has k numbers, so $n' \ge m_k$. By Step 1, $2s \ge n' \ge m_k$, so $s \ge \lceil \frac{m_k}{2} \rceil$. Hence $n = s^2 + n' \ge \lceil \frac{m_k}{2} \rceil^2 + m_k$.

Achieved. Let $t = \lceil \frac{m_k}{2} \rceil$ and $n = t^2 + m_k$. Since $m_k \le 2t < 2t + 1$, we get $n < (t+1)^2$, so the largest square at most n really is t^2 and the next number is m_k. Thus

$$m_{k+1} = \left\lceil \frac{m_k}{2} \right\rceil^2 + m_k.$$

Step 3 (compute).

$$\begin{aligned} &m_3 = 1^2 + 1 = 2, \quad m_4 = 1^2 + 2 = 3, \quad m_5 = 2^2 + 3 = 7, \\ &m_6 = 4^2 + 7 = 23, \quad m_7 = 12^2 + 23 = 167, \quad m_8 = 84^2 + 167 = 7223. \end{aligned}$$

Here $84 = \lceil \frac{167}{2} \rceil$ and $84^2 = 7056$.

Step 4 (check). Starting from 7223: $7223 - 84^2 = 167$, $167 - 12^2 = 23$, $23 - 4^2 = 7$, $7 - 2^2 = 3$, $3 - 1^2 = 2$, $2 - 1^2 = 1$, $1 - 1^2 = 0$. That is 8 numbers, so $N = 7223$ and its units digit is 3.

Watch out. Round *up*, not down. With $83 = \left\lfloor \frac{167}{2} \right\rfloor$ you would get $83^2 + 167 = 7056 = 84^2$, a perfect square whose sequence is just $7056, 0$. Also count carefully: miscounting by one in either direction — 7 numbers ($N = 167$) or 9 numbers ($N = 13{,}053{,}767$) — gives units digit 7, choice (D).

Answer: (B) 3

---

25 Counting & Probability

Source: 2015 AMC 10A #25

Let S be a square of side length 1. Two points are chosen independently at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is $\frac{a - b\pi}{c}$, where a, b, and c are positive integers with $\gcd(a, b, c) = 1$. What is $a + b + c$?

(A) 59 **(B)** 60 **(C)** 61 **(D)** 62 **(E)** 63

Key idea. Two points on the boundary of a square can be on the same side, on adjacent sides, or on opposite sides. Condition on which, and each case becomes a simple area problem in a unit square — the π in the answer hints that one case involves a circle.

Step 1. Which sides? Whatever side the first point is on, the second point is equally likely to be on any of the four sides. So it is on the same side with probability $\frac{1}{4}$, on an adjacent side with probability $\frac{2}{4} = \frac{1}{2}$, and on the opposite side with probability $\frac{1}{4}$. On its side, each point is uniformly placed.

Step 2. Same side. Let the positions along the side be u, v, independent and uniform on $[0, 1]$. The distance is $|u - v|$, and $|u - v| \geq \frac{1}{2}$ is the region of the unit square near the corners $(1, 0)$ and $(0, 1)$: two right triangles with legs $\frac{1}{2}$, total area $2 \cdot \frac{1}{8} = \frac{1}{4}$. Probability $\frac{1}{4}$.

Step 3. Adjacent sides. Measure each point's distance from the shared corner: u and v, independent and uniform on $[0, 1]$. The two sides are perpendicular, so the distance is $\sqrt{u^2 + v^2}$. It is less than $\frac{1}{2}$ inside the quarter disk $u^2 + v^2 < \frac{1}{4}$, which fits inside the unit square and has area $\frac{1}{4} \cdot \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{16}$. Probability $1 - \frac{\pi}{16}$.

Step 4. Opposite sides. The sides are parallel and 1 apart, so the distance is always at least $1 > \frac{1}{2}$. Probability 1.

Step 5. Combine.

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2}\left(1 - \frac{\pi}{16}\right) + \frac{1}{4} \cdot 1 = \frac{2}{32} + \frac{16}{32} - \frac{\pi}{32} + \frac{8}{32} = \frac{26 - \pi}{32}.$$

So $a = 26$, $b = 1$, $c = 32$, and $\gcd(26, 1, 32) = 1$ because $b = 1$. Hence $a + b + c = 26 + 1 + 32 = 59$.

Answer: (A) 59