AMC 10

PRACTICE EXAMINATION

SET 7

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 25 problems are taken from past AMC 10 contests (2000–2025) and appear in their original wording. Advanced level: each problem sits two or three places earlier than it did in its original contest, and problems 21–25 come from original positions 23–25. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**D**	2004 AMC 10A #4	Algebra
2	**E**	2011 AMC 10B #5	Number Theory
3	**B**	2007 AMC 10A #5	Algebra
4	**C**	2013 AMC 10A #7	Counting & Probability
5	**B**	2006 AMC 10B #8	Geometry
6	**E**	2010 AMC 10A #9	Number Theory
7	**C**	2023 AMC 10B #10	Counting & Probability
8	**E**	2005 AMC 10B #11	Algebra
9	**C**	2015 AMC 10A #11	Geometry
10	**E**	2005 AMC 10A #13	Algebra
11	**D**	2009 AMC 10B #14	Algebra
12	**E**	2023 AMC 10A #15	Geometry
13	**E**	2024 AMC 10A #15	Number Theory
14	**D**	2021 Fall AMC 10A #17	Geometry
15	**B**	2024 AMC 10B #18	Number Theory
16	**B**	2022 AMC 10B #18	Counting & Probability
17	**E**	2022 AMC 10A #20	Algebra
18	**E**	2010 AMC 10B #21	Counting & Probability
19	**D**	2015 AMC 10B #22	Geometry
20	**B**	2019 AMC 10B #22	Counting & Probability
21	**B**	2008 AMC 10A #23	Counting & Probability
22	**C**	2008 AMC 10B #24	Geometry
23	**C**	2017 AMC 10B #24	Geometry
24	**C**	2018 AMC 10B #25	Algebra
25	**A**	2020 AMC 10A #25	Counting & Probability

1 Algebra

Source: 2004 AMC 10A #4

What is the value of x if $|x-1| = |x-2|$?

(A) $-\frac{1}{2}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** $\frac{3}{2}$ **(E)** 2

Key idea. Read $|x-a|$ as "the distance from x to a on the number line." The equation then says x is the same distance from 1 as from 2.

The only point on the number line equally far from 1 and from 2 is the midpoint of the segment joining them:

$$x = \frac{1+2}{2} = \frac{3}{2}.$$

Check: $\left|\frac{3}{2}-1\right| = \frac{1}{2}$ and $\left|\frac{3}{2}-2\right| = \frac{1}{2}$. The two sides agree.

Another way. Both sides are non-negative, so squaring keeps the equation equivalent: $(x-1)^2 = (x-2)^2$, that is, $x^2 - 2x + 1 = x^2 - 4x + 4$. The x^2 terms cancel, leaving $2x = 3$, so $x = \frac{3}{2}$.

Watch out. Dropping the bars and writing $x - 1 = x - 2$ gives $-1 = -2$, which has no solution. Absolute values force two cases; the one that works is $x - 1 = -(x - 2)$.

Answer: (D) $\frac{3}{2}$

2 Number Theory

Source: 2011 AMC 10B #5

In multiplying two positive integers a and b, Ron reversed the digits of the two-digit number a. His erroneous product was 161. What is the correct value of the product of a and b?

(A) 116 **(B)** 161 **(C)** 204 **(D)** 214 **(E)** 224

Key idea. Ron's wrong product is (reversed a) $\times\, b = 161$, so the reversed number must be a factor of 161. Factor 161 and look for a two-digit factor.

Step 1. Factor: $161 = 7 \cdot 23$. Since 7 and 23 are both prime, the factors of 161 are $1, 7, 23, 161$.

Step 2. The reversed number has two digits, and the only two-digit factor is 23. So Ron used 23 in place of a, and $b = 161 \div 23 = 7$.

Step 3. The true a is 23 with its digits reversed: $a = 32$. The correct product is

$$32 \cdot 7 = 224.$$

(If a ended in 0, its reversal would be a one-digit number such as 1 or 7, giving $a = 10$ or $a = 70$ and a correct product of 1610 — not a choice.)

Watch out. Answering 161 (B) — that is Ron's *erroneous* product, not the one the question asks for.

Answer: (E) 224

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3 Algebra

Source: 2007 AMC 10A #5

A school store sells 7 pencils and 8 notebooks for \$4.15. It also sells 5 pencils and 3 notebooks for \$1.77. How much do 16 pencils and 10 notebooks cost?

(A) \$4.76 **(B)** \$5.84 **(C)** \$6.00 **(D)** \$6.16 **(E)** \$6.32

Key idea. Two unknown prices and two purchases: write a system of two linear equations, solve it, then price the new order. Working in cents keeps everything whole.

Let a pencil cost p cents and a notebook n cents. Then

$$7p + 8n = 415, \qquad 5p + 3n = 177.$$

Step 1. Eliminate n: multiply the first equation by 3 and the second by 8: $21p + 24n = 1245$ and $40p + 24n = 1416$. Subtracting, $19p = 171$, so $p = 9$.

Step 2. Substitute into $5p + 3n = 177$: $45 + 3n = 177$, so $3n = 132$ and $n = 44$. Check: $7(9) + 8(44) = 63 + 352 = 415$. ✓

Step 3. The new order costs $16(9) + 10(44) = 144 + 440 = 584$ cents, which is \$5.84.

Answer: (B) \$5.84

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4 Counting & Probability

Source: 2013 AMC 10A #7

A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 16

Key idea. English is forced, so only three more courses are chosen. "At least one" is easiest by complement: count everything, then remove the choices with *no* mathematics.

Step 1. With English fixed, choose the other 3 courses from the 5 remaining (Algebra, Geometry, History, Art, Latin). That gives $\binom{5}{3} = 10$ programs.

Step 2. The mathematics courses are Algebra and Geometry. A program with no mathematics must take its 3 extra courses from History, Art, Latin — there is exactly $\binom{3}{3} = 1$ such program.

Step 3. So the number of valid programs is $10 - 1 = 9$.

Watch out. "Pick one math course (2 ways), then any 2 of the other 4 ($\binom{4}{2} = 6$ ways)" gives $2 \cdot 6 = 12$, choice (D). This counts every program containing *both* Algebra and Geometry twice — there are 3 of them, and $12 - 3 = 9$.

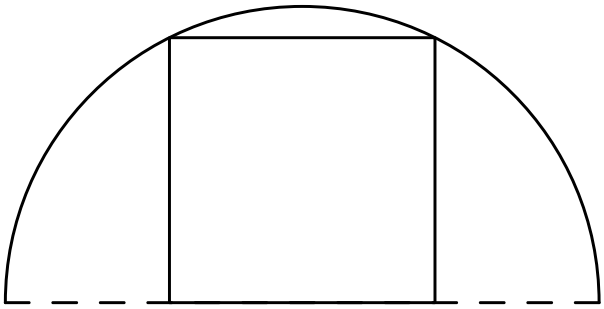
Answer: (C) 9

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5 Geometry

Source: 2006 AMC 10B #8

A square of area 40 is inscribed in a semicircle as shown. What is the area of the semicircle?



(A) 20π **(B)** 25π **(C)** 30π **(D)** 40π **(E)** 50π

Key idea. Join the center of the semicircle to a top corner of the square: that segment is a radius, and it is the hypotenuse of a right triangle whose legs you know.

Let the square have side s, so $s^2 = 40$. Let O be the center of the semicircle (the midpoint of the diameter).

The two top corners of the square lie on the arc, so both are at distance r from O. They are also at the same height, so they are the same horizontal distance from O. Hence O is the midpoint of the bottom side of the square, and each top corner is $\frac{s}{2}$ to the side of O and s above it.

By the Pythagorean theorem,

$$r^2 = \left(\frac{s}{2}\right)^2 + s^2 = \frac{40}{4} + 40 = 10 + 40 = 50.$$

The semicircle has half the area of the full circle: $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi \cdot 50 = 25\pi$.

Watch out. Stopping at $\pi r^2 = 50\pi$ gives (E) — that is the whole circle. The question asks for the semicircle, so halve it.

Answer: (B) 25π

6 Number Theory

Source: 2010 AMC 10A #9

A *palindrome*, such as 83438, is a number that remains the same when its digits are reversed. The numbers x and $x + 32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of x?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

Key idea. Adding only 32 turns a three-digit number into a four-digit one, so $x+32$ must be squeezed into a very short range just above 1000.

Because x has three digits, $x \leq 999$, so $x + 32 \leq 1031$. Because $x + 32$ has four digits, $x + 32 \geq 1000$. So

$$1000 \leq x + 32 \leq 1031.$$

A four-digit palindrome starting with 1 must end with 1 and has the form $1aa1$. The smallest are 1001 and then 1111, and only 1001 lies in our range. Hence $x + 32 = 1001$, so $x = 969$.

Check: 969 reads the same backwards, so it really is a three-digit palindrome. Its digit sum is $9+6+9 = 24$.

Answer: (E) 24

7 Counting & Probability

Source: 2023 AMC 10B #10

You are playing a game. A 2×1 rectangle covers two adjacent squares (oriented either horizontally or vertically) of a 3×3 grid of squares, but you are not told which two squares are covered. Your goal is to find at least one square that is covered by the rectangle. A "turn" consists of you guessing a square, after which you are told whether that square is covered by the hidden rectangle. What is the minimum number of turns you need to ensure that at least one of your guessed squares is covered by the rectangle?

(A) 3 **(B)** 5 **(C)** 4 **(D)** 8 **(E)** 6

Key idea. Think about the unlucky case where every answer is "no". Then your guesses form a fixed list, and the list works exactly when every possible position of the rectangle contains at least one guessed square. So you need a lower bound (positions no small list can cover) and a construction (a list that covers them all).

Step 1: at least 4 turns. Here are four rectangle positions that share no square: the left two squares of the top row, the top two squares of the right column, the right two squares of the bottom row, and the bottom two squares of the left column (a pinwheel around the centre). The rectangle could be in any of them, so each needs its own guessed square. Hence 3 guesses can never be enough.

Step 2: 4 turns suffice. Colour the grid like a checkerboard with the corners black. The four white squares are the midpoints of the edges. Any 2×1 rectangle covers two adjacent squares, which always have different colours, so it covers exactly one white square. Guess the four edge midpoints: one of them must be covered.

So the minimum is 4.

Watch out. Guessing the whole middle row (3 turns) catches every vertical rectangle, but a horizontal rectangle in the top row avoids it — so (A) 3 fails.

Answer: (C) 4

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8 Algebra

Source: 2005 AMC 10B #11

The first term of a sequence is 2005. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2005th term of the sequence?

(A) 29 **(B)** 55 **(C)** 85 **(D)** 133 **(E)** 250

Key idea. 2005 is far too many terms to compute, so the sequence must start repeating. Compute a few terms and look for a cycle.

Step 1. Compute the first terms:

$$2^3+0^3+0^3+5^3=133,\quad 1^3+3^3+3^3=55,\quad 5^3+5^3=250,\quad 2^3+5^3+0^3=133.$$

So the sequence is 2005, 133, 55, 250, 133, 55, 250, …, and from the 2nd term on it repeats the cycle 133, 55, 250 of length 3.

Step 2. The 2nd, 5th, 8th, … terms are 133; these are the terms whose position leaves remainder 2 when divided by 3. Likewise positions with remainder 0 give 55 and positions with remainder 1 (from the 4th term on) give 250.

Step 3. Since $2005 = 3 \cdot 668 + 1$ leaves remainder 1, the 2005th term is 250.

Watch out. The first term 2005 is not part of the cycle. If you pretend the cycle starts at term 1 (term 1 is 133, term 2 is 55, term 3 is 250, …), then remainder 1 points to 133, choice (D).

Answer: (E) 250

9 Geometry

Source: 2015 AMC 10A #11

> The ratio of the length to the width of a rectangle is $4:3$. If the rectangle has diagonal of length d, then the area may be expressed as kd^2 for some constant k. What is k?
>
> **(A)** $\frac{2}{7}$ **(B)** $\frac{3}{7}$ **(C)** $\frac{12}{25}$ **(D)** $\frac{16}{25}$ **(E)** $\frac{3}{4}$

Key idea. A ratio $4:3$ means the sides are $4x$ and $3x$ for a single unknown x. Writing *both* the area and d in terms of x lets x cancel, which is exactly why the problem promises a constant k.

Step 1. Let the length be $4x$ and the width $3x$, with $x > 0$.

Step 2. The diagonal is the hypotenuse of a right triangle with legs $4x$ and $3x$. By the Pythagorean theorem,

$$d^2 = (4x)^2 + (3x)^2 = 16x^2 + 9x^2 = 25x^2,$$

so $d = 5x$. (This is the familiar 3–4–5 right triangle, scaled by x.)

Step 3. The area is $4x \cdot 3x = 12x^2$.

Step 4. We want $12x^2 = kd^2 = k \cdot 25x^2$. Dividing both sides by $25x^2$ (which is not zero),

$$k = \frac{12x^2}{25x^2} = \frac{12}{25}.$$

Another way. Pick a convenient size: the rectangle with sides 4 and 3 has diagonal 5 and area 12, so $k = \frac{12}{5^2} = \frac{12}{25}$. Because every $4:3$ rectangle is a scaled copy of this one, and both the area and d^2 scale by the same factor, k is the same for all of them.

Answer: (C) $\frac{12}{25}$

10 Algebra

Source: 2005 AMC 10A #13

How many positive integers n satisfy the following condition:

$$(130n)^{50} > n^{100} > 2^{200}?$$

(A) 0 **(B)** 7 **(C)** 12 **(D)** 65 **(E)** 125

Key idea. Every exponent is a multiple of 50, so take 50th roots of the whole chain; the huge powers collapse to a simple quadratic inequality.

Step 1. All three quantities are positive, and for positive numbers $a > b$ exactly when $a^{50} > b^{50}$. Since $n^{100} = (n^2)^{50}$ and $2^{200} = (2^4)^{50} = 16^{50}$, the condition is the same as

$$130n > n^2 > 16.$$

Step 2. Right half: $n^2 > 16$ with n positive means $n > 4$, so $n \geq 5$.

Step 3. Left half: $130n > n^2$. Divide by the positive number n: $130 > n$, so $n \leq 129$.

Step 4. The integers $n = 5, 6, \ldots, 129$ all work, and there are $129 - 5 + 1 = 125$ of them.

Watch out. Counting 5 to 129 as $129 - 5 = 124$ is the classic fence-post slip; remember to add 1 when counting both ends.

Answer: (E) 125

11 Algebra

Source: 2009 AMC 10B #14

On Monday, Millie puts a quart of seeds, 25% of which are millet, into a bird feeder. On each successive day she adds another quart of the same mix of seeds without removing any seeds that are left. Each day the birds eat only 25% of the millet in the feeder, but they eat all of the other seeds. On which day, just after Millie has placed the seeds, will the birds find that more than half the seeds in the feeder are millet?

(A) Tuesday **(B)** Wednesday **(C)** Thursday **(D)** Friday **(E)** Saturday

Key idea. Track only the millet. The other seeds never pile up — the birds eat them all — so right after each filling there is exactly $\frac{3}{4}$ quart of non-millet, and "more than half millet" just means "more than $\frac{3}{4}$ quart of millet."

Step 1. Each quart brings $\frac{1}{4}$ quart of millet and $\frac{3}{4}$ quart of other seeds. Before the next filling the birds eat 25% of the millet, so $\frac{3}{4}$ of it is still there the next day.

Step 2. Millet just after each filling (in quarts):

$$\text{Mon: } \tfrac{1}{4}, \quad \text{Tue: } \tfrac{3}{4} \cdot \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{7}{16}, \quad \text{Wed: } \tfrac{3}{4} \cdot \tfrac{7}{16} + \tfrac{1}{4} = \tfrac{37}{64},$$

$$\text{Thu: } \tfrac{3}{4}\cdot\tfrac{37}{64}+\tfrac{1}{4}=\tfrac{175}{256}, \quad \text{Fri: } \tfrac{3}{4}\cdot\tfrac{175}{256}+\tfrac{1}{4}=\tfrac{781}{1024}.$$

In general, on the nth day (Monday is $n=1$) the millet is $\tfrac{1}{4}\big(1+\tfrac{3}{4}+\cdots+(\tfrac{3}{4})^{n-1}\big)=1-(\tfrac{3}{4})^n$.

Step 3. We need millet $>\tfrac{3}{4}$, i.e. $(\tfrac{3}{4})^n<\tfrac{1}{4}$. On Thursday ($n=4$), $(\tfrac{3}{4})^4=\tfrac{81}{256}>\tfrac{64}{256}=\tfrac{1}{4}$: millet $\tfrac{175}{256}\approx 0.68$, not enough. On Friday ($n=5$), $(\tfrac{3}{4})^5=\tfrac{243}{1024}<\tfrac{256}{1024}=\tfrac{1}{4}$: millet $\tfrac{781}{1024}\approx 0.76>0.75$.

So the first such day is the fifth day, Friday.

Watch out. Monday is day 1, not day 0. Counting five days *after* Monday lands on Saturday, choice (E).

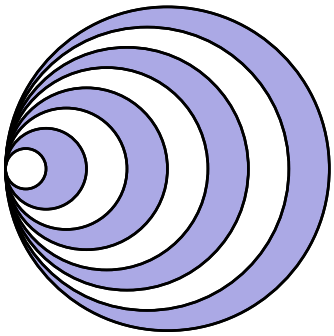
Answer: (D) Friday

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12 Geometry

Source: 2023 AMC 10A #15

An even number of circles are nested, starting with a radius of 1 and increasing by 1 each time, all sharing a common point. The region between every other circle is shaded, starting with the region inside the circle of radius 2 but outside the circle of radius 1. An example showing 8 circles is displayed below. What is the least number of circles needed to make the total shaded area at least 2023π?



(A) 46 **(B)** 48 **(C)** 56 **(D)** 60 **(E)** 64

Key idea. The common point is a distraction: each circle lies inside the next, so the area of a ring is just the difference of two disk areas, wherever the centres are. Writing one ring's area as a difference of squares makes the whole sum collapse into $1+2+3+\cdots$.

Step 1: one ring. A circle of radius r encloses area πr^2. The shaded ring between radii $2k-1$ and $2k$ has area

$$\pi\big((2k)^2-(2k-1)^2\big)=\pi\big(2k+(2k-1)\big)\cdot\big(2k-(2k-1)\big)=\pi\big(2k+(2k-1)\big).$$

For example, the first ring has area $(2+1)\pi=3\pi$ and the second $(4+3)\pi=7\pi$.

Step 2: all rings. With N circles (N even), the shaded rings are between radii 1 and 2, 3 and 4, …, $N-1$ and N. Adding the areas from Step 1,

$$\text{shaded area}=\pi(1+2+3+4+\cdots+N)=\pi\cdot\frac{N(N+1)}{2}.$$

Check with the picture: 8 circles give $\pi \cdot \frac{8\cdot 9}{2} = 36\pi$, and indeed $3+7+11+15 = 36$.

Step 3: the inequality. We need $\frac{N(N+1)}{2} \geq 2023$, that is $N(N+1) \geq 4046$. Since $63 \cdot 64 = 4032$ is close, try the even values near it:

$$N = 62: \ \frac{62\cdot 63}{2} = 1953 < 2023, \qquad N = 64: \ \frac{64\cdot 65}{2} = 2080 \geq 2023.$$

The area grows with N, so every even $N \leq 62$ falls short, and the least number of circles is 64.

Watch out. With $N = 2n$ circles the area is $(2n^2+n)\pi$, and $2n^2+n \geq 2023$ first holds at $n = 32$. That 32 counts shaded *rings*; the question asks for circles, which is twice as many.

Answer: (E) 64

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13 Number Theory

Source: 2024 AMC 10A #15

Let M be the greatest integer such that both $M+1213$ and $M+3773$ are perfect squares. What is the units digit of M?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 8

Key idea. Two perfect squares that differ by a fixed amount: write them as x^2 and y^2 and factor the difference $x^2 - y^2 = (x-y)(x+y)$. A fixed product with a large answer means the two factors should be as far apart as possible.

Step 1. Write $M+3773 = x^2$ and $M+1213 = y^2$ with $x > y \geq 0$. Subtracting,

$$x^2 - y^2 = 3773 - 1213 = 2560, \qquad \text{so} \qquad (x-y)(x+y) = 2560.$$

Step 2. The factors $x-y$ and $x+y$ differ by $2y$, an even number, so they are both even or both odd. Their product 2560 is even, so both are even. Write $x-y = 2s$ and $x+y = 2t$; then $4st = 2560$, so $st = 640$, and $y = t-s$.

Step 3. $M = y^2 - 1213$ is largest when $y = t-s$ is largest. With $st = 640$ and $1 \leq s < t$, this happens for the smallest s, namely $s = 1$, $t = 640$, so $y = 639$ and $x = 641$. (Any other factor pair has $s \geq 2$, $t \leq 320$, and $y \leq 318$.)

Step 4. Then $M = 639^2 - 1213 = 408321 - 1213 = 407108$. Check: $M + 3773 = 410881 = 641^2$. The units digit of M is 8.

For the units digit alone you do not need the full product: 639^2 ends in 1 (since $9^2 = 81$), and $\ldots 1 - \ldots 3$ ends in 8.

Watch out. Taking the factor pair $1 \cdot 2560$ looks like it makes x and y far apart, but $x-y = 1$, $x+y = 2560$ gives $y = \frac{2559}{2}$, not an integer. The parity check in Step 2 is what rules it out.

Answer: (E) 8

14 Geometry

Source: 2021 Fall AMC 10A #17

An architect is building a structure that will place vertical pillars at the vertices of regular hexagon $ABCDEF$, which is lying horizontally on the ground. The six pillars will hold up a flat solar panel that will not be parallel to the ground. The heights of pillars at A, B, and C are 12, 9, and 10 meters, respectively. What is the height, in meters, of the pillar at E?

(A) 9 **(B)** $6\sqrt{3}$ **(C)** $8\sqrt{3}$ **(D)** 17 **(E)** $12\sqrt{3}$

Key idea. A flat panel means the height is a *linear* function of position, so the height above the midpoint of any segment is the average of the heights at its ends. The hexagon is full of shared midpoints, which lets you hop from A, B, C to the center and then to E.

Step 1. Over a flat plane, the height has the form $h = ux+vy+w$ in ground coordinates (x, y). Plugging in the midpoint of two points gives exactly the average of their two heights. So if two segments have the same midpoint, the sums of the heights at their ends are equal.

Step 2. Let O be the center of the hexagon. Triangles OAB and OBC are equilateral, so $ABCO$ is a rhombus. Its diagonals $\overline{AC}$ and $\overline{BO}$ bisect each other, so they share a midpoint. Therefore

$$h(A) + h(C) = h(B) + h(O) \quad \Longrightarrow \quad 12 + 10 = 9 + h(O) \quad \Longrightarrow \quad h(O) = 13.$$

Step 3. B and E are opposite vertices, so O is the midpoint of $\overline{BE}$. Hence $h(O)$ is the average of $h(B)$ and $h(E)$:

$$13 = \frac{9 + h(E)}{2} \quad \Longrightarrow \quad h(E) = 26 - 9 = 17.$$

Another way. Coordinates: put $A = (-1, 0)$, $B = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $E = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. From $h(C) - h(B) = u = 1$ and $h(A) = -u + w = 12$ we get $w = 13$; then $h(B) = -\frac{1}{2} + \frac{\sqrt{3}}{2}v + 13 = 9$ gives $\frac{\sqrt{3}}{2}v = -\frac{7}{2}$. So $h(E) = \frac{1}{2} - \frac{\sqrt{3}}{2}v + 13 = \frac{1}{2} + \frac{7}{2} + 13 = 17$.

Answer: (D) 17

15 Number Theory

Source: 2024 AMC 10B #18

> How many different remainders can result when the 100th power of an integer is divided by 125?
>
> **(A)** 1 **(B)** 2 **(C)** 5 **(D)** 25 **(E)** 125

Key idea. The exponent 100 is no accident: exactly 100 of the numbers $1, 2, \ldots, 125$ are not multiples of 5. That points to Euler's theorem, and splits the integers into two cases: multiples of 5 and the rest.

Case 1: $5 \mid n$. Then n^{100} is a multiple of 5^{100}, which is certainly a multiple of $125 = 5^3$. So the remainder is 0.

Case 2: $5 \nmid n$. Euler's theorem says that if n and m have no common factor greater than 1, then $n^{\varphi(m)}$ leaves remainder 1 when divided by m, where $\varphi(m)$ counts the numbers from 1 to m with no common factor greater than 1 with m. For $m = 125$, the numbers 1 to 125 include 25 multiples of 5, so $\varphi(125) = 125 - 25 = 100$. Hence n^{100} leaves remainder 1.

Count. Both remainders actually occur: 5^{100} leaves 0 and $1^{100} = 1$ leaves 1. So exactly 2 different remainders are possible.

Watch out. Euler's theorem only covers n not divisible by 5. Applying it to every integer gives the single remainder 1, choice (A).

Another way. Without Euler's theorem, raise to the fifth power twice. For $5 \nmid n$, Fermat's little theorem gives $n^4 = 1 + 5k$ for some integer k. By the binomial theorem,

$$n^{20} = (1 + 5k)^5 = 1 + 25k + 250k^2 + \cdots,$$

and every term after the 1 is a multiple of 25, so $n^{20} = 1 + 25j$. Again,

$$n^{100} = (1 + 25j)^5 = 1 + 125j + \cdots,$$

with every term after the 1 a multiple of 125. So n^{100} leaves remainder 1.

Answer: (B) 2

16 Counting & Probability

Source: 2022 AMC 10B #18

Consider systems of three linear equations with unknowns x, y, and z,

$$\begin{cases} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{cases}$$

where each of the coefficients is either 0 or 1 and the system has a solution other than $x = y = z = 0$. For example, one such system is

$$\begin{cases} 1x + 1y + 0z &= 0 \\ 0x + 1y + 1z &= 0 \\ 0x + 0y + 0z &= 0 \end{cases}$$

with a nonzero solution of $(x, y, z) = (1, -1, 1)$. How many such systems of equations are there? (The equations in a system need not be distinct, and two systems containing the same equations in a different order are considered different.)

(A) 302 **(B)** 338 **(C)** 340 **(D)** 343 **(E)** 344

Key idea. There are only $2^9 = 512$ coefficient patterns, so count the complement: the systems whose *only* solution is $x = y = z = 0$. Those are the systems whose three rows (a_i, b_i, c_i) are independent — no row can be built from the others — and that is a short list to check.

Step 1: total. Each of the 9 coefficients is 0 or 1, so there are $2^9 = 512$ systems, counting order of equations.

Step 2: when is 0 the only solution? Think of each equation as its row (a_i, b_i, c_i); the solutions of one equation with a nonzero row form a plane through the origin. If a row is $(0, 0, 0)$, or two rows are equal, the system really has at most two different nonzero rows, and two planes through the origin always share a line — so there is a nonzero solution. Thus a system with only the zero solution must have three *distinct nonzero* rows. There are 7 nonzero rows, giving $7 \cdot 6 \cdot 5 = 210$ ordered choices.

Step 3: dependent triples. Three distinct nonzero rows still give a nonzero solution when one row is the sum of the other two: for example, from $(0, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$ the system is really just $z = 0$, $y = 0$, and x is free. Checking all $\binom{7}{3} = 35$ unordered triples of the 7 nonzero rows, the dependent ones are exactly the 6 in which one row is the sum of the other two (writing 100 for the row $(1, 0, 0)$, and so on):

- the sum has two 1s: $\{100, 010, 110\}$, $\{100, 001, 101\}$, $\{010, 001, 011\}$;
- the sum is 111: $\{100, 011, 111\}$, $\{010, 101, 111\}$, $\{001, 110, 111\}$.

Each can be ordered in $3! = 6$ ways, giving 36 ordered dependent choices.

Step 4. Systems with only the zero solution: $210 - 36 = 174$. Systems with a nonzero solution: $512 - 174 = 338$.

Watch out. Counting only the "obvious" systems — those with a zero row or a repeated row — gives $512 - 210 = 302$, choice (A). The 36 systems of Step 3 have three different nonzero rows but still have a nonzero solution.

Answer: (B) 338

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17 Algebra

Source: 2022 AMC 10A #20

A four-term sequence is formed by adding each term of a four-term arithmetic sequence of positive integers to the corresponding term of a four-term geometric sequence of positive integers. The first three terms of the resulting four-term sequence are 57, 60, and 91. What is the fourth term of this sequence?

(A) 190 **(B)** 194 **(C)** 198 **(D)** 202 **(E)** 206

Key idea. Take differences twice. An arithmetic sequence has constant differences, so its *second* difference is 0; whatever second difference the sum shows must come entirely from the geometric part. Then use that everything is a positive integer.

Step 1. Set up. Let the arithmetic sequence be $a,\ a+d,\ a+2d,\ a+3d$ and the geometric one $b,\ br,\ br^2,\ br^3$. Then

$$a+b=57, \qquad a+d+br=60, \qquad a+2d+br^2=91.$$

Step 2. Differences. Subtracting consecutive equations:

$$d+b(r-1)=3, \qquad d+br(r-1)=31.$$

Subtracting again removes d: $br(r-1)-b(r-1)=b(r-1)^2=28$.

Step 3. Integrality. Let $t=b(r-1)=br-b$, an integer since b and br are. Multiply $b(r-1)^2=28$ by b: $t^2=28b$. Then $7 \mid t^2$ forces $7 \mid t$, and t^2 even forces t even, so $t=14u$ for a nonzero integer u. Now $196u^2=28b$ gives $b=7u^2$, and $r-1=\frac{t}{b}=\frac{2}{u}$.

Step 4. Cases. Since $a=57-b>0$, $7u^2<57$, so $u\in\{\pm1,\pm2\}$.

- $u=-1$ or $u=-2$: $r=-1$ or $r=0$, impossible for positive terms.
- $u=2$: $b=28$, $r=2$, $a=29$, $d=3-t=3-28=-25$, so $a+2d=-21<0$. Rejected.
- $u=1$: $b=7$, $r=3$, $a=50$, $d=3-14=-11$.

The sequences are $50, 39, 28, 17$ and $7, 21, 63, 189$; their sums are $57, 60, 91, 206$, matching the given terms.

The fourth term is $17+189=206$.

Watch out. Checking only that a and b are positive lets $u=2$ slip through. Always confirm *every* term of the arithmetic sequence is positive — here $29, 4, -21, \ldots$ fails.

Answer: (E) 206

18 Counting & Probability

Source: 2010 AMC 10B #21

A palindrome between 1000 and 10,000 is chosen at random. What is the probability that it is divisible by 7?

(A) $\frac{1}{10}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{1}{7}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{5}$

Key idea. A four-digit palindrome is fixed by its first two digits, so write it as $\overline{abba}$ and expand in place value. The coefficient that appears, 1001, is the famous $7 \cdot 11 \cdot 13$ — that is why the divisor 7 in the question is friendly.

Step 1: count the palindromes. Every palindrome between 1000 and 10,000 has four digits and looks like $\overline{abba}$, where a is the thousands digit and b the hundreds digit. The first digit a can be 1 through 9 (9 choices) and b can be 0 through 9 (10 choices). The last two digits are then forced, so there are $9 \cdot 10 = 90$ palindromes.

Step 2: expand. By place value,

$$\overline{abba} = 1000a + 100b + 10b + a = 1001a + 110b.$$

Step 3: test divisibility by 7. Since $1001 = 7 \cdot 143$, the term $1001a$ is always a multiple of 7. So $\overline{abba}$ is divisible by 7 exactly when $110b$ is. Now $110 = 7 \cdot 15 + 5$ is not a multiple of 7, and 7 is prime, so 7 divides $110b$ exactly when 7 divides b. Among the digits 0–9, that means $b = 0$ or $b = 7$.

Step 4: count and divide. Any a from 1 to 9 works with either b, giving $9 \cdot 2 = 18$ palindromes divisible by 7 (for example $1001 = 7 \cdot 143$ and $1771 = 7 \cdot 253$). The probability is

$$\frac{18}{90} = \frac{1}{5}.$$

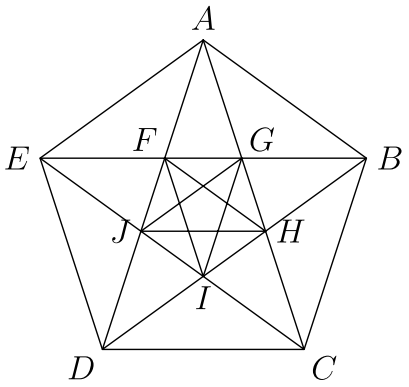
Watch out. Guessing that "about one number in seven is divisible by 7" gives $\frac{1}{7}$, choice (C). Palindromes are not spread out like ordinary integers: every one of them has the form $1001a + 110b$, and only the digit b decides divisibility by 7, so the share is 2 digits out of 10.

Answer: (E) $\frac{1}{5}$

19 Geometry

Source: 2015 AMC 10B #22

In the figure shown below, $ABCDE$ is a regular pentagon and $AG = 1$. What is $FG + JH + CD$?



(A) 3 **(B)** $12 - 4\sqrt{5}$ **(C)** $\frac{5+2\sqrt{5}}{3}$ **(D)** $1+\sqrt{5}$ **(E)** $\frac{11+11\sqrt{5}}{10}$

Key idea. A regular pentagon with its diagonals is full of 36°–72°–72° isosceles triangles, and any two of them are similar. Name the one unknown length $b = FG$, find two such triangles whose sides involve b, and their similarity gives an equation for b.

Step 1: the angles. Each interior angle of a regular pentagon is 108°. Triangle ABC is isosceles ($AB = BC$) with apex angle 108°, so $\angle BAC = \angle BCA = 36°$; in the same way, the two diagonals from any vertex split its 108° angle into three angles of 36°. For example $\angle ABE = \angle EBD = \angle DBC = 36°$, so $\angle ABD = 72°$.

Step 2: lengths along AC. The figure is symmetric about the line through A and the midpoint of $\overline{CD}$, and the inner pentagon $FGHIJ$ is regular. So $GH = FG = b$, and $HC = AG = 1$ (by the symmetry about the line through B). Hence $AH = 1 + b$.

In $\triangle ABG$, $\angle GAB = \angle GBA = 36°$, so $GB = GA = 1$.

Step 3: two similar triangles. In $\triangle ABH$: $\angle BAH = 36°$, $\angle ABH = \angle ABD = 72°$, so $\angle AHB = 72°$. It is isosceles with $AB = AH = 1 + b$, and its base is BH.

In $\triangle BGH$: $\angle GBH = \angle EBD = 36°$ and $\angle BHG = \angle BHA = 72°$, so $\angle BGH = 72°$. It is isosceles with $BH = BG = 1$, and its base is $GH = b$.

Both are 36°–72°–72° triangles, so base/leg is the same:

$$\frac{b}{1} = \frac{1}{1+b} \quad \Longrightarrow \quad b^2 + b - 1 = 0 \quad \Longrightarrow \quad b = \frac{\sqrt{5}-1}{2},$$

taking the positive root of the quadratic because b is a length.

Step 4: the three lengths. $FG = b$. Side $CD = AB = 1 + b$. For JH: by symmetry $AJ = AH$ and $AF = AG$, so $\overline{FG} \parallel \overline{JH}$ and $\triangle AFG \sim \triangle AJH$ with ratio $\frac{AH}{AG} = 1+b$. Thus $JH = b(1+b) = b^2+b = 1$.

Step 5. Adding,

$$FG + JH + CD = b + 1 + (1 + b) = 2 + 2b = 2 + (\sqrt{5} - 1) = 1 + \sqrt{5}.$$

Answer: (D) $1 + \sqrt{5}$

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20 Counting & Probability

Source: 2019 AMC 10B #22

Raashan, Sylvia, and Ted play the following game. Each starts with \$1. A bell rings every 15 seconds, at which time each of the players who currently has money simultaneously chooses one of the other two players independently and at random and gives \$1 to that player. What is the probability that after the bell has rung 2019 times, each player will have \$1?

(For example, Raashan and Ted may each decide to give \$1 to Sylvia, and Sylvia may decide to give her dollar to Ted, at which point Raashan will have \$0, Sylvia will have \$2, and Ted will have \$1, and that is the end of the first round of play. In the second round Raashan has no money to give, but Sylvia and Ted might choose each other to give their \$1 to, and the holdings will be the same at the end of the second round.)

(A) $\frac{1}{7}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

Key idea. 2019 rounds is far too many to track, so the number is a signal: the answer probably does not depend on the history. List the few money distributions that can occur and compute, from *each* one, the chance that the next ring produces $(\$1, \$1, \$1)$. If that chance is the same from every state, it is the answer.

Step 1. Which states can occur? There are always \$3 in total, and nobody can ever end a round holding all \$3. Suppose some player P did. If P had money at the ring, P gave a dollar to someone else, who then does not have \$0 (that dollar makes up for any dollar the recipient gave away). If P had \$0, the other two held all \$3 but each gave away only \$1, so at least \$1 stayed with them. Hence after every ring the holdings are either $(1, 1, 1)$ or some arrangement of $(2, 1, 0)$.

Step 2. Starting from $(1, 1, 1)$. All three players give, each choosing one of two people, so there are $2^3 = 8$ equally likely ways. Everyone ends with \$1 exactly when everyone *receives* exactly one dollar, which means the dollars go around in a circle: either Raashan→Sylvia→Ted→Raashan, or the reverse. That is 2 of the 8 ways, so the probability is $\frac{2}{8} = \frac{1}{4}$.

Step 3. Starting from $(2, 1, 0)$. Call the players X (with \$2), Y (with \$1) and Z (with \$0). Only X and Y give, so there are $2 \cdot 2 = 4$ equally likely ways:

- $X \to Y$, $Y \to X$: still X=2, Y=1, Z=0;
- $X \to Y$, $Y \to Z$: now X=1, Y=1, Z=1;
- $X \to Z$, $Y \to X$: now X=2, Y=0, Z=1;
- $X \to Z$, $Y \to Z$: now X=1, Y=0, Z=2.

Exactly one of the four gives $(1,1,1)$, so the probability is again $\frac{1}{4}$.

Step 4. Conclude. Whatever the holdings are after 2018 rings, the 2019th ring produces $(\$1,\$1,\$1)$ with probability $\frac{1}{4}$. So the overall probability is $\frac{1}{4}$.

Watch out. There are 7 possible states — $(1,1,1)$ and the 6 arrangements of $(2,1,0)$ — but they are not equally likely. Assuming they are gives $\frac{1}{7}$, choice (A). In fact $(1,1,1)$ has probability $\frac{1}{4}$ and each of the other six has $\frac{1}{8}$.

Answer: (B) $\frac{1}{4}$

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21 Counting & Probability

Source: 2008 AMC 10A #23

Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 **(B)** 40 **(C)** 60 **(D)** 160 **(E)** 320

Key idea. Decide the fate of each element separately. Because the union is all of S, every element is in the first subset only, the second only, or both. That turns the problem into counting labellings — and then fixing the overcount from order.

Step 1: the shared elements. Exactly two elements lie in both subsets. Choose them in $\binom{5}{2} = 10$ ways.

Step 2: the other three. Each of the remaining 3 elements must be in the union but not in the intersection, so it lies in exactly one of the two subsets. If for now we call the subsets "first" and "second", each element has 2 choices, giving $2^3 = 8$ ways.

So there are $10 \cdot 8 = 80$ *ordered* pairs (first subset, second subset).

Step 3: remove the order. The two subsets are never equal: each of the 3 non-shared elements is in one but not the other. So every unordered choice $\{A, B\}$ was counted exactly twice, once as (A, B) and once as (B, A). The answer is $\frac{80}{2} = 40$.

For example, choosing $\{a, b\}$ as the intersection and putting c in the first subset, d, e in the second gives $\{a, b, c\}$ and $\{a, b, d, e\}$; the swapped labelling gives the same two subsets.

Watch out. Stopping at 80 ignores the phrase "order …does not matter"; 80 is not a choice, which is the hint to divide by 2.

Answer: (B) 40

22 Geometry

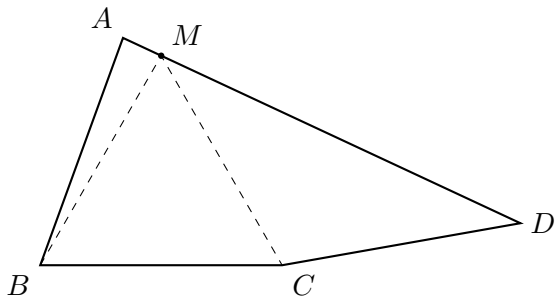
Source: 2008 AMC 10B #24

Quadrilateral $ABCD$ has $AB = BC = CD$, $\angle ABC = 70^\circ$, and $\angle BCD = 170^\circ$. What is the degree measure of $\angle BAD$?

(A) 75 **(B)** 80 **(C)** 85 **(D)** 90 **(E)** 95

Key idea. Three equal sides $AB = BC = CD$ and two angles, 70° and 170°, that are each "60° plus a nice amount" suggest building an *equilateral triangle on* $\overline{BC}$. Its third vertex M is as far from B as A is, and as far from C as D is, so two isosceles triangles appear whose angles we can compute.

Step 1. The auxiliary point. Let M be the point on the same side of $\overline{BC}$ as A with $\triangle BMC$ equilateral, so $BM = MC = BC$ and every angle of $\triangle BMC$ is 60°. Since $70^\circ > 60^\circ$ and $170^\circ > 60^\circ$, ray BM lies inside $\angle ABC$ and ray CM lies inside $\angle BCD$.



Step 2. Triangle ABM. Here $AB = BC = BM$, so $\triangle ABM$ is isosceles with apex angle $\angle ABM = 70^\circ - 60^\circ = 10^\circ$. Its base angles are equal:

$$\angle BAM = \angle BMA = \frac{180^\circ - 10^\circ}{2} = 85^\circ.$$

Step 3. Triangle MCD. Here $MC = BC = CD$, so $\triangle MCD$ is isosceles with apex angle $\angle MCD = 170^\circ - 60^\circ = 110^\circ$. Its base angles are

$$\angle CMD = \angle CDM = \frac{180^\circ - 110^\circ}{2} = 35^\circ.$$

Step 4. M lies on $\overline{AD}$. Around M, the rays MA, MB, MC, MD come in this order (see the figure), so the three adjacent angles between them add up to

$$\angle AMB + \angle BMC + \angle CMD = 85^\circ + 60^\circ + 35^\circ = 180^\circ.$$

So rays MA and MD point in opposite directions: A, M, D are collinear, with M between A and D.

Step 5. Since M is on segment $\overline{AD}$, the angle at A is $\angle BAD = \angle BAM = 85^\circ$.

Check. Then $\angle ADC = \angle MDC = 35^\circ$, and the four angles of $ABCD$ add to $85^\circ + 70^\circ + 170^\circ + 35^\circ = 360^\circ$, as they must.

Answer: (C) 85

23 Geometry

Source: 2017 AMC 10B #24

> The vertices of an equilateral triangle lie on the hyperbola $xy = 1$, and a vertex of this hyperbola is the centroid of the triangle. What is the square of the area of the triangle?
>
> **(A)** 48 **(B)** 60 **(C)** 108 **(D)** 120 **(E)** 169

Key idea. In an equilateral triangle the centroid is also the circumcenter. So the three vertices are points of $xy = 1$ that are all the *same distance* from the hyperbola's vertex — and distances from $(1, 1)$ to points $(t, \frac{1}{t})$ have a very tidy formula.

Step 1. Set-up. The vertices of $xy = 1$ are $(1, 1)$ and $(-1, -1)$; by the symmetry $(x, y) \mapsto (-x, -y)$ we may take the centroid to be $G = (1, 1)$. Since the triangle is equilateral, G is its circumcenter: all three vertices lie at the same distance R from G.

Step 2. Distance from G. For a point $(t, \frac{1}{t})$, let $s = t + \frac{1}{t}$. Then

$$(t-1)^2 + \left(\tfrac{1}{t} - 1\right)^2 = t^2 + \tfrac{1}{t^2} - 2s + 2 = (s^2 - 2) - 2s + 2 = s(s-2).$$

On the branch $t > 0$ we have $s \geq 2$, and $s(s-2)$ grows as s grows; on the branch $t < 0$ we have $s \leq -2$, and $s(s-2)$ grows as s decreases, starting from $(-2)(-4) = 8$. So for a given R^2 each branch has at most one value of s, and a value of s gives the two points $(t, \frac{1}{t})$ and $(\frac{1}{t}, t)$ — mirror images across $y = x$ (one point if $t = \pm 1$).

Step 3. Which three points? So the circle of radius R about G meets the hyperbola in at most two mirror pairs, one on each branch, and our triangle uses three of these points: a whole pair from one branch and one point from the other. Use the centroid: the x-coordinates and the y-coordinates each add to 3.

- Whole pair $(p, \frac{1}{p}), (\frac{1}{p}, p)$ with $p > 0$, plus $(q, \frac{1}{q})$ with $q < 0$: the sums are $p + \frac{1}{p} + q$ and $p + \frac{1}{p} + \frac{1}{q}$, so $q = \frac{1}{q}$, giving $q = -1$.
- Whole pair on the negative branch plus $(p, \frac{1}{p})$ with $p > 0$: the same comparison forces $p = 1$, i.e. the vertex $(1, 1) = G$ itself — impossible, since a vertex is not the center.

So one vertex is $(-1, -1)$, and $R^2 = 2^2 + 2^2 = 8$. On the positive branch, $s(s-2) = 8$ gives $s = 4$ (the x-sum $s + (-1) = 3$ agrees), so the other vertices are $(2 + \sqrt{3},\ 2 - \sqrt{3})$ and $(2 - \sqrt{3},\ 2 + \sqrt{3})$. Each side of this triangle has squared length 24, so it really is equilateral.

Step 4. Area. The center splits the triangle into three triangles with two sides R and angle 120° between them:

$$\text{Area} = 3 \cdot \frac{1}{2} R^2 \sin 120^\circ = \frac{3}{2} \cdot 8 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}.$$

(Equivalently, side$^2 = 24$ and $\frac{\sqrt{3}}{4} \cdot 24 = 6\sqrt{3}$.) The square of the area is $(6\sqrt{3})^2 = 108$.

Answer: (C) 108

24 Algebra

Source: 2018 AMC 10B #25

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2 + 10{,}000\lfloor x \rfloor = 10{,}000x$?

(A) 197 **(B)** 198 **(C)** 199 **(D)** 200 **(E)** 201

Key idea. Move the floor term to the other side: $x^2 = 10{,}000\,(x - \lfloor x \rfloor)$. The right side is 10,000 times the *fractional part* of x, which lies between 0 and 1. That pins x to a bounded range and suggests checking one unit interval $[a, a+1)$ at a time — picture a very flat parabola $y = \frac{x^2}{10{,}000}$ crossing the sawtooth $y = x - \lfloor x \rfloor$ once per tooth.

Step 1. Bound x. Let $a = \lfloor x \rfloor$. The equation becomes $x^2 = 10{,}000(x - a)$. Since $0 \le x - a < 1$, we get $0 \le x^2 < 10{,}000$, so $-100 < x < 100$. Hence a is one of the integers $-100, -99, \ldots, 99$.

Step 2. One interval at a time. Fix such an integer a and look for solutions with $a \le x < a + 1$, that is, zeros of

$$g(x) = 10{,}000(x - a) - x^2.$$

For $a \le x < y < a + 1$ we have

$$g(y) - g(x) = 10{,}000(y - x) - (y^2 - x^2) = (y - x)\big(10{,}000 - (x + y)\big) > 0,$$

because $x + y < 200$. So g is increasing on the interval and has *at most one* zero there.

Step 3. When is there a zero? At the left end, $g(a) = -a^2$; as x approaches $a + 1$, $g(x)$ approaches $10{,}000 - (a + 1)^2$.

- If $a = 0$: $g(0) = 0$, so $x = 0$ is a solution.
- If $a \ne 0$ and $(a + 1)^2 < 10{,}000$: g starts negative and ends positive, so it crosses 0 exactly once strictly inside the interval.
- If $(a+1)^2 \ge 10{,}000$: since g is increasing, $g(x) < 10{,}000 - (a+1)^2 \le 0$ throughout, so there is no solution.

Step 4. Count. Among $a = -100, \ldots, 99$, the condition $(a + 1)^2 < 10{,}000$ means $-99 \le a + 1 \le 99$, i.e. $-100 \le a \le 98$. It fails only for $a = 99$ (and $a = 0$ also gives exactly one solution). So there is exactly one solution for each of

$$a = -100, -99, \ldots, 98,$$

which is $98 - (-100) + 1 = 199$ solutions.

Watch out. For $a = 99$ the equation $x^2 - 10{,}000x + 990{,}000 = 0$ factors as $(x - 100)(x - 9900) = 0$, and $x = 100$ looks like a solution "at the end of the interval." But $\lfloor 100 \rfloor = 100$, not 99, and indeed $100^2 + 10{,}000 \cdot 100 \ne 10{,}000 \cdot 100$. Counting it gives 200, choice (D).

Answer: (C) 199

25 Counting & Probability

Source: 2020 AMC 10A #25

> Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?
>
> **(A)** $\frac{7}{36}$ **(B)** $\frac{5}{24}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{17}{72}$ **(E)** $\frac{1}{4}$

Key idea. First work out how good each kind of reroll can possibly be — rerolling 0, 1, 2 or 3 dice. Comparing these few numbers tells you exactly which starting rolls make "reroll two" the best move, and then it is a counting problem.

Step 1: The value of each option.

- *Reroll none:* wins (probability 1) only if the sum is already 7.
- *Reroll one*, keeping two dice with sum t: the new die must show $7-t$, which is possible if and only if $t \le 6$. The probability is $\frac{1}{6}$ if $t \le 6$, and 0 otherwise.
- *Reroll two*, keeping one die showing d: the two new dice must total $7-d$. Two dice make a sum m (for $2 \le m \le 7$) in $m-1$ ways (and a sum of 1 is impossible), so this happens in $6-d$ of the 36 outcomes: $5, 4, 3, 2, 1, 0$ for $d = 1, \ldots, 6$.
- *Reroll all three:* the number of ways three dice total 7 is $\binom{6}{2} = 15$, so the probability is $\frac{15}{216} = \frac{5}{72} = \frac{2.5}{36}$.

Step 2: When is "reroll two" best? Rerolling two dice wins with probability at most $\frac{5}{36}$, which is less than $\frac{1}{6} = \frac{6}{36}$. So if some pair of dice sums to 6 or less, rerolling the third die beats every two-dice reroll. (A roll with sum 7 also has such a pair, and Jason simply keeps it.) Hence Jason rerolls two dice only if *every* pair sums to at least 7 — that is, the two smallest dice sum to at least 7.

In that case rerolling none or one die gives probability 0. Rerolling two is best when he keeps the smallest die a, giving $\frac{6-a}{36}$, and this beats $\frac{2.5}{36}$ exactly when $6-a \ge 3$, i.e. $a \le 3$. So he rerolls exactly two dice precisely when the smallest die a is at most 3 and the two smallest dice sum to at least 7.

Step 3: Count. Write the roll in order $a \le b \le c$ and count ordered rolls:

- $a = 1$, $b \ge 6$: $(1,6,6)$, giving 3 rolls.
- $a = 2$, $b \ge 5$: $(2,5,5)$, $(2,5,6)$, $(2,6,6)$, giving $3+6+3 = 12$.
- $a = 3$, $b \ge 4$: $(3,4,4)$, $(3,4,5)$, $(3,4,6)$, $(3,5,5)$, $(3,5,6)$, $(3,6,6)$, giving $3+6+6+3+6+3 = 27$.

That is $3 + 12 + 27 = 42$ of the 216 rolls, so the probability is $\dfrac{42}{216} = \dfrac{7}{36}$.

Watch out. Forgetting to compare with rerolling all three dice lets in the rolls with smallest die 4, 5 or 6 as well; that gives $\frac{69}{216} = \frac{23}{72}$, which is not a choice. A smallest die of 4 leaves only $\frac{2}{36}$, worse than the $\frac{2.5}{36}$ from starting over.

Answer: (A) $\frac{7}{36}$