AMC 10

PRACTICE EXAMINATION

SET 8

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 25 problems are taken from past AMC 10 contests (2000–2025) and appear in their original wording. Advanced level: each problem sits two or three places earlier than it did in its original contest, and problems 21–25 come from original positions 23–25. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**E**	2004 AMC 10A #3	Algebra
2	**B**	2010 AMC 10B #5	Number Theory
3	**D**	2013 AMC 10A #6	Counting & Probability
4	**E**	2018 AMC 10A #7	Number Theory
5	**B**	2015 AMC 10A #8	Algebra
6	**D**	2005 AMC 10B #9	Counting & Probability
7	**C**	2014 AMC 10B #10	Number Theory
8	**C**	2023 AMC 10A #11	Geometry
9	**B**	2016 AMC 10B #11	Algebra
10	**D**	2007 AMC 10B #12	Algebra
11	**B**	2022 AMC 10B #14	Counting & Probability
12	**D**	2020 AMC 10A #14	Algebra
13	**D**	2021 Fall AMC 10B #15	Geometry
14	**D**	2021 Fall AMC 10A #16	Algebra
15	**B**	2010 AMC 10A #18	Counting & Probability
16	**B**	2009 AMC 10A #19	Geometry
17	**D**	2012 AMC 10A #19	Algebra
18	**C**	2011 AMC 10B #20	Geometry
19	**C**	2013 AMC 10B #22	Counting & Probability
20	**C**	2025 AMC 10B #23	Number Theory
21	**D**	2012 AMC 10B #23	Geometry
22	**C**	2025 AMC 10A #24	Counting & Probability
23	**D**	2021 AMC 10A #24	Geometry
24	**C**	2008 AMC 10A #25	Geometry
25	**A**	2017 AMC 10A #25	Number Theory

1 Algebra

Source: 2004 AMC 10A #3

Alicia earns \$20 per hour, of which 1.45% is deducted to pay local taxes. How many cents per hour of Alicia's wages are used to pay local taxes?

(A) 0.0029 **(B)** 0.029 **(C)** 0.29 **(D)** 2.9 **(E)** 29

Key idea. The question asks for *cents*, so convert the wage to cents first; then a percent is just a multiplication by a decimal.

Step 1. One dollar is 100 cents, so \$20 is $20 \times 100 = 2000$ cents per hour.

Step 2. A percent means "out of 100", so $1.45\% = \dfrac{1.45}{100} = 0.0145$.

Step 3. The local tax is

$$0.0145 \times 2000 = 29$$

cents per hour.

As a check: 1% of 2000 cents is 20 cents, so 1.45% should be a little under 30 cents — and 29 is.

Watch out. Working in dollars gives $0.0145 \times 20 = 0.29$, which is the tax in *dollars*. Reporting that number as cents picks (C) 0.29; it must still be multiplied by 100.

Answer: (E) 29

2 Number Theory

Source: 2010 AMC 10B #5

A month with 31 days has the same number of Mondays and Wednesdays. How many of the seven days of the week could be the first day of this month?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Key idea. Four full weeks use up 28 days and contain exactly four of every weekday. Only the 3 leftover days decide which weekdays appear five times.

Since $31 = 4 \cdot 7 + 3$, the weekdays that occur five times are exactly the first three days of the month. The month has equally many Mondays and Wednesdays when those three days contain *both* of them or *neither* of them.

Go through the seven possible first days:

- Monday: Mon, Tue, Wed — both. **Works.**
- Tuesday: Tue, Wed, Thu — Wednesday only. Fails.
- Wednesday: Wed, Thu, Fri — Wednesday only. Fails.

- Thursday: Thu, Fri, Sat — neither. **Works.**
- Friday: Fri, Sat, Sun — neither. **Works.**
- Saturday: Sat, Sun, Mon — Monday only. Fails.
- Sunday: Sun, Mon, Tue — Monday only. Fails.

So the first day can be Monday, Thursday, or Friday: 3 days.

Watch out. It is easy to look only for the case where both days get a fifth occurrence (Monday) and forget the "neither" cases, Thursday and Friday.

Answer: (B) 3

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3 Counting & Probability

Source: 2013 AMC 10A #6

Joey and his five brothers are ages $3, 5, 7, 9, 11$, and 13. One afternoon two of his brothers whose ages sum to 16 went to the movies, two brothers younger than 10 went to play baseball, and Joey and the 5-year-old stayed home. How old is Joey?

(A) 3 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

Key idea. List the pairs of ages that add to 16 and rule them out one at a time using who stayed home and who went to baseball.

The pairs with sum 16 are $(3, 13)$, $(5, 11)$ and $(7, 9)$.

The 5-year-old stayed home, so the movie pair is not $(5, 11)$.

If 7 and 9 went to the movies, the brothers younger than 10 who are left would be only the 3-year-old and the 5-year-old — but the 5-year-old stayed home, so there would be just one boy for baseball, not two. So the movie pair is not $(7, 9)$ either.

Hence 3 and 13 went to the movies. The baseball pair must be two of the remaining ages $5, 7, 9, 11$ that are under 10 and are not the 5-year-old: that is 7 and 9. The only age left is 11, and that is Joey.

Check: movies $3 + 13 = 16$, baseball $7, 9 < 10$, home 5 and 11 — all six ages used once.

Answer: (D) 11

4 Number Theory

Source: 2018 AMC 10A #7

For how many (not necessarily positive) integer values of n is the following value an integer?

$$4000 \cdot \left(\frac{2}{5}\right)^n$$

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

Key idea. Split everything into prime factors. The expression is an integer exactly when no prime ends up with a negative exponent.

Since $4000 = 2^5 \cdot 5^3$,

$$4000 \cdot \left(\frac{2}{5}\right)^n = 2^5 \cdot 5^3 \cdot \frac{2^n}{5^n} = 2^{5+n} \cdot 5^{3-n}.$$

This holds for negative n too: then $\left(\frac{2}{5}\right)^n = \left(\frac{5}{2}\right)^{-n}$, and the formula still gives the right powers.

A number $2^a \cdot 5^b$ is an integer exactly when $a \geq 0$ and $b \geq 0$ (a negative exponent leaves a 2 or a 5 in the denominator that nothing can cancel). So we need

$$5 + n \geq 0 \quad \text{and} \quad 3 - n \geq 0, \quad \text{that is,} \quad -5 \leq n \leq 3.$$

The integers from -5 to 3 are $-5, -4, \ldots, 3$: there are $3 - (-5) + 1 = 9$ of them.

Watch out. The words "not necessarily positive" are there for a reason. Counting only $n = 0, 1, 2, 3$ gives 4, choice (B); and computing $3 - (-5) = 8$ without the "+1" for counting both endpoints gives 8, choice (D).

Answer: (E) 9

5 Algebra

Source: 2015 AMC 10A #8

Two years ago Pete was three times as old as his cousin Claire. Two years before that, Pete was four times as old as Claire. In how many years will the ratio of their ages be $2 : 1$?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

Key idea. Name the two *current* ages, turn each sentence into an equation, and solve. Note that "two years before that" means four years ago.

Step 1. Let Pete be p and Claire be c years old now. Two years ago: $p - 2 = 3(c - 2)$, so $p = 3c - 4$. Four years ago: $p - 4 = 4(c - 4)$, so $p = 4c - 12$.

Step 2. Setting the two expressions for p equal gives $3c - 4 = 4c - 12$, so $c = 8$ and $p = 3 \cdot 8 - 4 = 20$. Check: two years ago they were 18 and 6 (three times), four years ago 16 and 4 (four times).

Step 3. In y years the ratio is $2:1$ when $20+y=2(8+y)=16+2y$, so $y=4$. Then Pete is 24 and Claire is 12.

Answer: (B) 4

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6 Counting & Probability

Source: 2005 AMC 10B #9

> One fair die has faces $1,1,2,2,3,3$ and another has faces $4,4,5,5,6,6$. The dice are rolled and the numbers on the top faces are added. What is the probability that the sum will be odd?
>
> **(A)** $\frac{1}{3}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{2}{3}$

Key idea. A sum is odd exactly when one number is odd and the other is even, so only the *parity* of each die matters.

Step 1. The first die has odd faces $1,1,3,3$ — four of six — so it is odd with probability $\frac{4}{6}=\frac{2}{3}$ and even with probability $\frac{1}{3}$.

Step 2. The second die has odd faces $5,5$ — two of six — so it is odd with probability $\frac{1}{3}$ and even with probability $\frac{2}{3}$.

Step 3. The two ways to get an odd sum are (odd, even) and (even, odd):

$$\frac{2}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot\frac{1}{3}=\frac{4}{9}+\frac{1}{9}=\frac{5}{9}.$$

Watch out. Counting only the case "first die odd, second die even" gives $\frac{4}{9}$, choice B. Assuming each die is odd half the time, as on an ordinary die, gives $\frac{1}{2}$, choice C.

Answer: (D) $\frac{5}{9}$

7 Number Theory

Source: 2014 AMC 10B #10

In the addition shown below A, B, C, and D are distinct digits. How many different values are possible for D?

$$\begin{array}{r} ABBCB \\ +\ BCADA \\ \hline DBDDD \end{array}$$

(A) 2 **(B)** 4 **(C)** 7 **(D)** 8 **(E)** 9

Key idea. Work column by column, but start with the units column and the leftmost column together: both add the same two digits A and B, and that pins down every carry.

Step 1. The units column adds $B + A$ and writes D, so $A + B$ is D or $D + 10$. The leftmost column adds $A + B$ (plus any carry) and writes D with *nothing* carried into a sixth digit, so $A + B \leq D \leq 9$. Hence $A + B = D$, and the units column carries nothing.

Step 2. The tens column now adds $C + D$ with no carry and writes D. So $C + D$ is D or $D + 10$; since $C \leq 9$, it must be $C + D = D$, giving $C = 0$ and no carry. The remaining columns check out: hundreds $B + A = D$, thousands $B + 0 = B$, and the leftmost $A + B = D$.

Step 3. So the puzzle works exactly when $C = 0$ and $D = A + B$, with A, B, D distinct and different from 0. A and B lead five-digit numbers, so they are nonzero, and they must differ; the smallest sum is $1 + 2 = 3$. Every D from 3 to 9 works, using $A = 1$, $B = D - 1$ (for instance $12202 + 20131 = 32333$ for $D = 3$). That is 7 values.

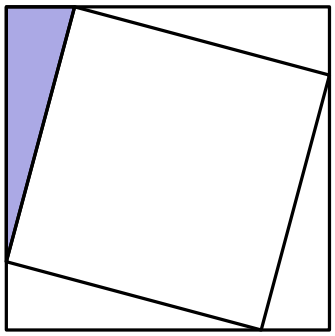
Watch out. Forgetting that A and B must be distinct lets $A = B = 1$ give $D = 2$, and you count 8 values, choice (D).

Answer: (C) 7

8 Geometry

Source: 2023 AMC 10A #11

A square of area 2 is inscribed in a square of area 3, creating four congruent triangles, as shown below. What is the ratio of the shorter leg to the longer leg in the shaded right triangle?



(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $2-\sqrt{3}$ **(D)** $\sqrt{3}-\sqrt{2}$ **(E)** $\sqrt{2}-1$

Key idea. Name the legs a and b. One side of the big square gives $a+b$, and the hypotenuse is a side of the small square, giving a^2+b^2. The ratio $\frac{a}{b}$ then comes from $\frac{a}{b}+\frac{b}{a}$, which uses exactly these two facts.

Step 1. Each side of the outer square is split into a short leg a and a long leg b of two corner triangles, so $a+b=\sqrt{3}$. The hypotenuse of each triangle is a side of the inner square, so $a^2+b^2=\left(\sqrt{2}\right)^2=2$.

Step 2. Square the first equation: $a^2+2ab+b^2=3$. Subtracting $a^2+b^2=2$ leaves $2ab=1$, so $ab=\frac{1}{2}$.

Step 3. Let $t=\frac{a}{b}$ with $0<t<1$. Then

$$t+\frac{1}{t}=\frac{a^2+b^2}{ab}=\frac{2}{1/2}=4,$$

so $t^2-4t+1=0$ and $t=2\pm\sqrt{3}$. Since $t<1$, we take $t=2-\sqrt{3}\approx 0.268$.

Another way. Find the legs themselves. Since $a+b=\sqrt{3}$ and $ab=\frac{1}{2}$, the legs are the roots of $t^2-\sqrt{3}\,t+\frac{1}{2}=0$:

$$t=\frac{\sqrt{3}\pm\sqrt{3-2}}{2}=\frac{\sqrt{3}\pm 1}{2}.$$

The ratio of the shorter to the longer is

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}.$$

Answer: (C) $2-\sqrt{3}$

9 Algebra

Source: 2016 AMC 10B #11

Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's garden?

(A) 256 **(B)** 336 **(C)** 384 **(D)** 448 **(E)** 512

Key idea. Count posts per side, remembering that each corner post belongs to two sides. Then convert posts to lengths by counting the *gaps* between posts.

Step 1. Let a short side have s posts and a long side ℓ posts, corners included. Adding the four sides counts each of the 4 corner posts twice, so

$$2\ell + 2s - 4 = 20.$$

Step 2. The long side has twice as many posts: $\ell = 2s$. Substituting, $4s + 2s - 4 = 20$, so $6s = 24$, giving $s = 4$ and $\ell = 8$.

Check: $2 \cdot 8 + 2 \cdot 4 - 4 = 20$ posts. Good.

Step 3. A row of n posts has $n - 1$ gaps between them. So a short side has 3 gaps and is $3 \cdot 4 = 12$ yards long; a long side has 7 gaps and is $7 \cdot 4 = 28$ yards long.

Step 4. The area is $12 \cdot 28 = 336$ square yards.

Watch out. Multiplying the number of posts (instead of gaps) by 4 gives sides 16 and 32, and area 512, choice (E). Four posts in a row enclose only three spaces.

Answer: (B) 336

10 Algebra

Source: 2007 AMC 10B #12

> Tom's age is T years, which is also the sum of the ages of his three children. His age N years ago was twice the sum of their ages then. What is $\frac{T}{N}$?
>
> **(A)** 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Key idea. Write both ages as they were N years ago. The key detail is that *each* of the three children was N years younger.

N years ago Tom was $T - N$. Each child was N years younger, so the children's ages added up to $T - 3N$.

The condition says Tom's age then was twice the children's total then:

$$T - N = 2(T - 3N) = 2T - 6N.$$

Move the T's to one side and the N's to the other: $6N - N = 2T - T$, so $T = 5N$. Therefore $\dfrac{T}{N} = 5$.

Watch out. Subtracting only N from the children's total (instead of $3N$) gives $T - N = 2(T - N)$, which forces $T = N$ — a ratio of 1, which is not a choice. Three children means three people aging.

Answer: (D) 5

11 Counting & Probability

Source: 2022 AMC 10B #14

> Suppose that S is a subset of $\{1, 2, 3, \cdots, 25\}$ such that the sum of any two (not necessarily distinct) elements of S is never an element of S. What is the maximum number of elements S may contain?
>
> **(A)** 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Key idea. A "maximum" question needs two halves: an example that reaches the number, and a proof that nothing beats it. For the proof, look at the *largest* element m of S: whenever i and $m - i$ are both in S, their sum m is in S too, which is forbidden — so the numbers below m pair up and each pair contributes at most one element.

Step 1 (13 is possible). Take $S = \{13, 14, \ldots, 25\}$, which has $25 - 13 + 1 = 13$ elements. Any two of them (even the same one twice) add to at least $13 + 13 = 26 > 25$, so no sum lands back in S.

Step 2 (13 is the most). Let m be the largest element of S. Split the numbers $1, 2, \ldots, m - 1$ into pairs $\{i,\ m - i\}$ with $i < m - i$; if m is even, the middle number $\frac{m}{2}$ is left over on its own.

If both numbers of a pair were in S, then $i + (m - i) = m$ would be a sum of two elements lying in S — not allowed. So each pair holds at most one element of S. If m is even, $\frac{m}{2} + \frac{m}{2} = m$, so $\frac{m}{2} \notin S$ either.

Step 3. There are $\left\lfloor \frac{m-1}{2} \right\rfloor$ pairs, so $|S| \le \left\lfloor \frac{m-1}{2} \right\rfloor + 1$, the $+1$ counting m itself. Since $m \le 25$, this is at most $\left\lfloor \frac{24}{2} \right\rfloor + 1 = 13$.

Together, the maximum is 13.

Another way. Another set of size 13 is all the odd numbers $\{1, 3, 5, \ldots, 25\}$: the sum of two odd numbers is even, so it is never in the set. Both examples meet the bound from Step 3 exactly.

Answer: (B) 13

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12 Algebra

Source: 2020 AMC 10A #14

Real numbers x and y satisfy $x + y = 4$ and $x \cdot y = -2$. What is the value of

$$x + \frac{x^3}{y^2} + \frac{y^3}{x^2} + y?$$

(A) 360 **(B)** 400 **(C)** 420 **(D)** 440 **(E)** 480

Key idea. The expression is symmetric in x and y, and we only know $x + y$ and xy — so rewrite everything in terms of those. Putting the two fractions over the common denominator $x^2y^2 = (xy)^2$ is the natural first move.

Step 1. The numbers x and y are the roots of $t^2 - 4t - 2 = 0$, namely $2 \pm \sqrt{6}$; both are real and nonzero, so the fractions make sense. Combining the fractions,

$$\frac{x^3}{y^2} + \frac{y^3}{x^2} = \frac{x^5 + y^5}{x^2y^2} = \frac{x^5 + y^5}{(-2)^2} = \frac{x^5 + y^5}{4}.$$

Step 2. Build up the power sums from $x + y = 4$ and $xy = -2$:

$$x^2 + y^2 = (x + y)^2 - 2xy = 16 + 4 = 20,$$

$$x^3 + y^3 = (x + y)^3 - 3xy(x + y) = 64 + 24 = 88.$$

Step 3. Multiplying out, $(x^2 + y^2)(x^3 + y^3) = x^5 + y^5 + x^2y^3 + x^3y^2 = x^5 + y^5 + x^2y^2(x + y)$. Therefore

$$x^5 + y^5 = 20 \cdot 88 - 4 \cdot 4 = 1760 - 16 = 1744.$$

Step 4. The two fractions sum to $\frac{1744}{4} = 436$. Adding the remaining $x + y = 4$ gives $436 + 4 = 440$.

Another way. Let $S_k = x^k + y^k$. Since x and y satisfy $t^2 = 4t + 2$, we get $S_k = 4S_{k-1} + 2S_{k-2}$. Starting from $S_0 = 2$, $S_1 = 4$: $S_2 = 20$, $S_3 = 88$, $S_4 = 392$, $S_5 = 1744$, as before.

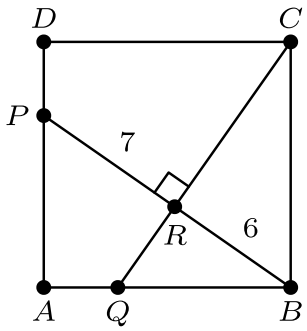
Watch out. Don't forget the outer x and y in the expression: the fractions alone give 436, which is not a choice. Also note $x^2y^2 = (-2)^2 = +4$, not -4.

Answer: (D) 440

13 Geometry

Source: 2021 Fall AMC 10B #15

In square $ABCD$, points P and Q lie on $\overline{AD}$ and $\overline{AB}$, respectively. Segments $\overline{BP}$ and $\overline{CQ}$ intersect at right angles at R, with $BR = 6$ and $PR = 7$. What is the area of the square?



(A) 85 **(B)** 93 **(C)** 100 **(D)** 117 **(E)** 125

Key idea. The right angle at R together with the right angles of the square creates similar right triangles. Name the side s and the length $AP = p$; the two given lengths then give two equations in s and p.

Step 1. Let s be the side of the square and $p = AP$. Since $BP = BR + RP = 6 + 7 = 13$, the right triangle PAB gives

$$s^2 + p^2 = 13^2 = 169.$$

Step 2. Compare triangles PAB and BRC. Both have a right angle ($\angle PAB = \angle BRC = 90^\circ$). Also $\angle APB = \angle RBC$, because $\overline{AD} \parallel \overline{BC}$ and $\overline{BP}$ crosses both (alternate angles). So the triangles are similar, with $P \leftrightarrow B$, $A \leftrightarrow R$, $B \leftrightarrow C$. Matching sides,

$$\frac{PA}{BR} = \frac{PB}{BC} \quad \Longrightarrow \quad \frac{p}{6} = \frac{13}{s} \quad \Longrightarrow \quad sp = 78.$$

Step 3. We want s^2, so look at s^2 and p^2: their sum is 169 and their product is $78^2 = 6084$. They are therefore the two roots of

$$t^2 - 169t + 6084 = 0, \quad \text{i.e.} \quad (t - 117)(t - 52) = 0.$$

So $\{s^2, p^2\} = \{117, 52\}$.

Step 4. P lies on side $\overline{AD}$, so $p = AP \le AD = s$. Hence s^2 is the larger root: the area is $s^2 = 117$ (and $p^2 = 52$).

Check: $s = 3\sqrt{13}$, $p = 2\sqrt{13}$, so $sp = 6 \cdot 13 = 78$ and $s^2 + p^2 = 117 + 52 = 169$.

Watch out. The other root, 52, is the configuration with P beyond D (then $p > s$). It is not a choice here, but always use the condition "P lies on $\overline{AD}$" to pick the root rather than guessing.

Answer: (D) 117

14 Algebra

Source: 2021 Fall AMC 10A #16

The graph of

$$f(x) = ||\lfloor x \rfloor| - |\lfloor 1 - x \rfloor||$$

is symmetric about which of the following? (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

(A) the y-axis **(B)** the line $x = 1$ **(C)** the origin

(D) the point $\left(\frac{1}{2}, 0\right)$ **(E)** the point $(1, 0)$

Key idea. Look at how x and $1-x$ appear in the formula: swapping them exchanges the two terms. That says $f(1-x) = -f(x)$, which is exactly point symmetry about $\left(\frac{1}{2}, 0\right)$.

Step 1. Replace x by $1 - x$. Since $1 - (1 - x) = x$,

$$f(1-x) = ||\lfloor 1 - x \rfloor| - |\lfloor x \rfloor|| = -f(x).$$

Step 2. So whenever (x, y) is on the graph, so is $(1 - x, -y)$. The midpoint of these two points is $\left(\frac{1}{2}, 0\right)$: the half-turn about $\left(\frac{1}{2}, 0\right)$ maps the graph onto itself.

Step 3. To see that the other choices fail, find the graph exactly. If x is not an integer and $n = \lfloor x \rfloor$, then $n < x < n + 1$ gives $-n < 1 - x < 1 - n$, so $\lfloor 1 - x \rfloor = -n$ and $f(x) = |n| - |-n| = 0$. At an integer n, $f(n) = |n| - |1 - n|$, which is $n - (n - 1) = 1$ for $n \geq 1$ and $-n - (1 - n) = -1$ for $n \leq 0$.

- y-axis: $f(1) = 1$ but $f(-1) = -1$.
- line $x = 1$: $f(0) = -1$ but $f(2) = 1$.
- origin: $(0, -1)$ is on the graph, so $(0, 1)$ would have to be too — impossible, since $f(0)$ has only one value.
- point $(1, 0)$: $(1, 1)$ is on the graph, so $(1, -1)$ would have to be — impossible for the same reason.

Watch out. Watch the minus sign. $f(1-x) = f(x)$ would mean mirror symmetry about the vertical line $x = \frac{1}{2}$; here we have $f(1 - x) = -f(x)$, which is symmetry about the *point* $\left(\frac{1}{2}, 0\right)$.

Answer: (D) $\left(\frac{1}{2}, 0\right)$

15 Counting & Probability

Source: 2010 AMC 10A #18

Bernardo randomly picks 3 distinct numbers from the set

$$\{1,2,3,4,5,6,7,8,9\}$$

and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set

$$\{1,2,3,4,5,6,7,8\}$$

and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

(A) $\frac{47}{72}$ **(B)** $\frac{37}{56}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{49}{72}$ **(E)** $\frac{39}{56}$

Key idea. The only difference between the two players is Bernardo's extra digit 9. Split on whether he picks it: if he does, he wins outright; if he does not, the two players are choosing from the *same* set, and symmetry does the rest.

Step 1: Bernardo picks a 9. He has $\binom{9}{3} = 84$ equally likely choices. Those containing 9 need two more digits from $\{1,\dots,8\}$: $\binom{8}{2} = 28$ ways. So this happens with probability $\frac{28}{84} = \frac{1}{3}$.

His number then starts with 9, while Silvia's starts with at most 8, so he always wins.

Step 2: Bernardo does not pick a 9. This has probability $1 - \frac{1}{3} = \frac{2}{3}$. Now both choose 3 digits from $\{1,\dots,8\}$, and both arrange them in descending order, so different choices give different numbers. The two players are in identical situations, so "Bernardo larger" and "Silvia larger" are equally likely.

They tie only if they pick the same set; Silvia matches Bernardo's set with probability $\frac{1}{\binom{8}{3}} = \frac{1}{56}$. Hence

$$P(\text{Bernardo larger}) = \frac{1 - \frac{1}{56}}{2} = \frac{55}{112}.$$

Step 3: combine the cases.

$$\frac{1}{3}\cdot 1 + \frac{2}{3}\cdot\frac{55}{112} = \frac{1}{3} + \frac{55}{168} = \frac{56}{168} + \frac{55}{168} = \frac{111}{168} = \frac{37}{56}.$$

Watch out. Forgetting that the two can tie in Step 2 and using $\frac{1}{2}$ there gives $\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{2} = \frac{2}{3}$, choice (C).

Answer: (B) $\frac{37}{56}$

16 Geometry

Source: 2009 AMC 10A #19

Circle A has radius 100. Circle B has an integer radius $r < 100$ and remains internally tangent to circle A as it rolls once around the circumference of circle A. The two circles have the same points of tangency at the beginning and end of circle B's trip. How many possible values can r have?

(A) 4 **(B)** 8 **(C)** 9 **(D)** 50 **(E)** 90

Key idea. Rolling without slipping means the distance travelled along circle A equals the length of circle B's rim that has touched it. So compare the two circumferences.

Circle A has circumference $2\pi \cdot 100 = 200\pi$, and circle B has circumference $2\pi r$. When B rolls once around A, the point of contact moves a distance 200π along A, and exactly the same length 200π of B's rim rolls through the contact point.

The point of tangency on A is back where it started after one trip. For the point on B to be the same too, the length 200π must be a whole number of trips around B:

$$\frac{200\pi}{2\pi r} = \frac{100}{r} \quad \text{must be a positive integer.}$$

So r must be a divisor of 100. The divisors of $100 = 2^2 \cdot 5^2$ are

$$1,\ 2,\ 4,\ 5,\ 10,\ 20,\ 25,\ 50,\ 100,$$

and the condition $r < 100$ rules out 100. That leaves 8 values.

Watch out. 100 has 9 divisors, and forgetting that $r < 100$ removes $r = 100$ gives 9, choice (C).

Answer: (B) 8

17 Algebra

Source: 2012 AMC 10A #19

Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8 : 00 AM, and all three always take the same amount of time to eat lunch.

On Monday the three of them painted 50% of a house, quitting at 4 : 00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2 : 12 PM. On Wednesday Paula worked by herself and finished the house by working until 7 : 12 P.M.

How long, in minutes, was each day's lunch break?

(A) 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 60

Key idea. Work = rate × working time, one equation per day. There are three unknowns (Paula's rate, the helpers' rate, the lunch length), so three days are just enough. The trick is to eliminate the rates first.

Step 1. Let the lunch break be L minutes, let Paula paint p percent of the house per minute, and let the two helpers together paint h percent per minute. Each day starts at 8:00 AM:

- Monday to 4:00 PM is 480 minutes;
- Tuesday to 2:12 PM is 372 minutes;
- Wednesday to 7:12 PM is 672 minutes.

Step 2. On Wednesday Paula *finished* the house, so she painted what was left: $100 - 50 - 24 = 26$ percent. The three days give

$$(p+h)(480-L) = 50, \qquad h(372-L) = 24, \qquad p(672-L) = 26.$$

Step 3. Tuesday plus Wednesday is $h(372-L) + p(672-L) = 50$, the same total as Monday, $p(480-L) + h(480-L) = 50$. Subtracting, the L's cancel:

$$p(480-672) + h(480-372) = 0 \quad \Longrightarrow \quad 108h = 192p \quad \Longrightarrow \quad h = \frac{16}{9}p.$$

Step 4. Monday becomes $\frac{25}{9}p(480-L) = 50$, so $p(480-L) = 18$. Dividing Wednesday's equation by this one removes p:

$$\frac{672-L}{480-L} = \frac{26}{18} = \frac{13}{9} \quad \Longrightarrow \quad 6048 - 9L = 6240 - 13L \quad \Longrightarrow \quad 4L = 192,$$

so $L = 48$.

Check. Then $p = \frac{18}{432} = \frac{1}{24}$ and $h = \frac{16}{9} \cdot \frac{1}{24} = \frac{2}{27}$. Monday: $\left(\frac{1}{24} + \frac{2}{27}\right) \cdot 432 = \frac{25}{216} \cdot 432 = 50$; Tuesday: $\frac{2}{27} \cdot 324 = 24$; Wednesday: $\frac{1}{24} \cdot 624 = 26$. All three days work.

Watch out. Wednesday's work is not 50%. Monday and Tuesday together already painted $50+24 = 74$ percent, so Paula only had 26% left to finish.

Answer: (D) 48

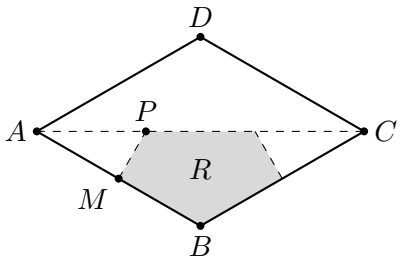
18 Geometry

Source: 2011 AMC 10B #20

Rhombus $ABCD$ has side length 2 and $\angle B = 120°$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R?

(A) $\frac{\sqrt{3}}{3}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\frac{2\sqrt{3}}{3}$ **(D)** $1+\frac{\sqrt{3}}{3}$ **(E)** 2

Key idea. "Closer to B than to A" is the side of the perpendicular bisector of $\overline{BA}$ that contains B. So R is cut out by three perpendicular bisectors — of $\overline{BA}$, $\overline{BC}$ and $\overline{BD}$ — and the 120° angle makes all of them easy to locate.



Step 1. In the rhombus, $\angle DAB = 180° - 120° = 60°$, so $\triangle ABD$ (with $AB = AD = 2$) is equilateral and the short diagonal is $BD = 2$. The diagonals of a rhombus are perpendicular bisectors of each other, so the perpendicular bisector of $\overline{BD}$ is the line AC. Thus R lies in $\triangle ABC$, on B's side of $\overline{AC}$.

Step 2. $\triangle ABC$ has $BA = BC = 2$ and $\angle B = 120°$, so its area is $\frac{1}{2}\cdot 2\cdot 2\cdot \sin 120° = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$. Its other two angles are each 30°.

Step 3. The perpendicular bisector of $\overline{BA}$ passes through the midpoint M of $\overline{BA}$ and meets $\overline{AC}$ at P. It cuts off right triangle AMP with $AM = 1$ and $\angle MAP = \angle BAC = 30°$, so $MP = AM\tan 30° = \frac{1}{\sqrt{3}}$ and its area is $\frac{1}{2}\cdot 1\cdot\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$. The bisector of $\overline{BC}$ cuts off a congruent triangle at C.

Step 4. Removing both corners, the area of R is $\sqrt{3} - 2\cdot\frac{\sqrt{3}}{6} = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$.

Check: by symmetry D also gets $\frac{2\sqrt{3}}{3}$, so A and C share the rest of the rhombus's area $2\sqrt{3}$, getting $\frac{\sqrt{3}}{3}$ each. The obtuse corners own more.

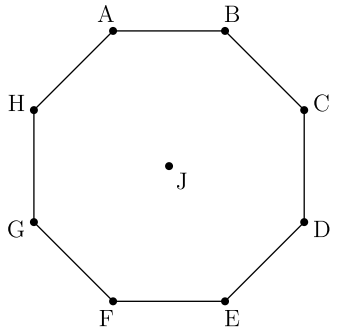
Watch out. Assuming the four vertices split the rhombus equally gives $\frac{1}{4}\cdot 2\sqrt{3} = \frac{\sqrt{3}}{2}$ — choice (B). They do not: the obtuse vertices B and D get twice as much as A and C.

Answer: (C) $\frac{2\sqrt{3}}{3}$

19 Counting & Probability

Source: 2013 AMC 10B #22

The regular octagon $ABCDEFGH$ has its center at J. Each of the vertices and the center are to be associated with one of the digits 1 through 9, with each digit used once, in such a way that the sums of the numbers on the lines AJE, BJF, CJG, and DJH are all equal. In how many ways can this be done?



(A) 384 **(B)** 576 **(C)** 1152 **(D)** 1680 **(E)** 3456

Key idea. All four lines pass through the center J, so "the four line sums are equal" really says "the four *opposite pairs* $\{A,E\},\{B,F\},\{C,G\},\{D,H\}$ have equal sums". Adding those four pair sums uses every digit except J — that pins down J.

Step 1. Which digits can sit at J? Each line sum is J plus the sum of an opposite pair, so all four pairs must have the same sum, say p. The four pairs use the eight digits other than J, whose total is $(1+2+\cdots+9)-J = 45-J$. Hence

$$4p = 45 - J,$$

so $45-J$ must be a multiple of 4. Checking $J = 1,\ldots,9$: $45-J = 44$, 40, 36 work for $J = 1, 5, 9$, and no other J does.

Step 2. The pairs are forced. Once J is chosen, $p = \frac{45-J}{4}$, and each remaining digit's partner must be p minus it:

- $J=1$, $p=11$: $\{2,9\},\{3,8\},\{4,7\},\{5,6\}$;
- $J=5$, $p=10$: $\{1,9\},\{2,8\},\{3,7\},\{4,6\}$;
- $J=9$, $p=9$: $\{1,8\},\{2,7\},\{3,6\},\{4,5\}$.

In each case these four pairs are the only way to split the remaining digits.

Step 3. Place the pairs. The four pairs can go to the four lines AJE, BJF, CJG, DJH in $4! = 24$ ways. On each line the two digits of its pair can be placed in 2 orders (e.g. $A=2, E=9$ or $A=9, E=2$), giving $2^4 = 16$ more choices. So each value of J gives

$$24 \cdot 16 = 384$$

arrangements.

Step 4. With three possible centers, the total is $3 \cdot 384 = 1152$.

Watch out. It is tempting to put the middle digit 5 in the center and stop there. That counts only one of the three cases and gives 384, choice (A).

Answer: (C) 1152

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20 Number Theory

Source: 2025 AMC 10B #23

A rectangular grid of squares has 141 rows and 91 columns. Each square has room for two numbers. Horace and Vera each fill in the grid by putting the numbers from 1 through $141 \times 91 = 12{,}831$ into the squares. Horace fills the grid horizontally: he puts 1 through 91 in order from left to right into row 1, puts 92 through 182 into row 2 in order from left to right, and continues similarly through row 141. Vera fills the grid vertically: she puts 1 through 141 in order from top to bottom into column 1, then 142 through 282 into column 2 in order from top to bottom, and continues similarly through column 91. How many squares get two copies of the same number?

(A) 7 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 19

Key idea. Write down a formula for the number each person puts in row i, column j, and set the two equal. Counting rows and columns from 0 instead of 1 makes the formulas, and the equation, much cleaner.

Step 1: the formulas. Let the square be in row i and column j, with $1 \le i \le 141$ and $1 \le j \le 91$. Horace has filled $i-1$ full rows of 91 before reaching row i, so he writes $91(i-1)+j$. Vera has filled $j-1$ full columns of 141, so she writes $141(j-1)+i$.

Step 2: the equation. Put $r = i-1$ and $c = j-1$, so $0 \le r \le 140$ and $0 \le c \le 90$. The numbers agree when

$$91r + (c+1) = 141c + (r+1) \iff 90r = 140c \iff 9r = 14c.$$

Step 3: solve in integers. Since $\gcd(9,14) = 1$, 9 must divide c and 14 must divide r: write $c = 9t$, $r = 14t$ for an integer $t \ge 0$. The limits require $9t \le 90$ and $14t \le 140$, i.e. $t \le 10$ both times.

So $t = 0, 1, \ldots, 10$ — that is 11 squares, from the top-left corner ($t = 0$, both write 1) to the bottom-right corner ($t = 10$, both write 12,831).

Watch out. Counting t from 1 to 10 (or computing $90 \div 9 = 10$) misses the corner $t = 0$ and gives 10, choice (B). Always check whether the first square counts.

Answer: (C) 11

21 Geometry

Source: 2012 AMC 10B #23

A solid tetrahedron is sliced off a solid wooden unit cube by a plane passing through two nonadjacent vertices on one face and one vertex on the opposite face not adjacent to either of the first two vertices. The tetrahedron is discarded and the remaining portion of the cube is placed on a table with the cut surface face down. What is the height of this object?

(A) $\frac{\sqrt{3}}{3}$ **(B)** $\frac{2\sqrt{2}}{3}$ **(C)** 1 **(D)** $\frac{2\sqrt{3}}{3}$ **(E)** $\sqrt{2}$

Key idea. Pin the cube down with coordinates to see exactly which corner is cut off. Then the height comes from the space diagonal through that corner: it is perpendicular to the cut, so the height is "whole diagonal minus the part that was removed." The removed part is the height of a tetrahedron, which the volume formula gives quickly.

Step 1. Which piece is cut off? Place the cube with vertices at (0 or 1, 0 or 1, 0 or 1). On the bottom face take the nonadjacent vertices $X = (1, 0, 0)$ and $Y = (0, 1, 0)$. On the top face, $(1, 0, 1)$ is adjacent to X and $(0, 1, 1)$ is adjacent to Y, so the third vertex is $(0, 0, 1)$ or $(1, 1, 1)$. These two cases are mirror images of each other, so take $Z = (0, 0, 1)$.

Now X, Y, Z are the three neighbours of the corner $P = (0, 0, 0)$, so the plane slices off the tetrahedron $PXYZ$. Let $Q = (1, 1, 1)$ be the vertex opposite P.

Step 2. Height of the discarded tetrahedron. Its base is the right triangle PXY with area $\frac{1}{2}$, and its height is $PZ = 1$, so its volume is $\frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$.

Seen another way, its base is the cut face XYZ, an equilateral triangle with side $\sqrt{2}$ (each side is a face diagonal) and area $\frac{\sqrt{3}}{4}\left(\sqrt{2}\right)^2 = \frac{\sqrt{3}}{2}$. If h is the distance from P to this face, then

$$\frac{1}{3} \cdot \frac{\sqrt{3}}{2} \cdot h = \frac{1}{6} \quad \Longrightarrow \quad h = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Step 3. The diagonal is perpendicular to the cut. P is at distance 1 from each of X, Y, Z, and Q is at distance $\sqrt{2}$ from each of them. A point equally far from the three vertices of a triangle lies on the line through the triangle's centre perpendicular to its plane. Both P and Q lie on that line, so the space diagonal $\overline{PQ}$, of length $\sqrt{3}$, is perpendicular to the cut face.

Step 4. The height. The cut plane is $x + y + z = 1$, and on the remaining side a point's distance from it is $\frac{x+y+z-1}{\sqrt{3}}$, which is largest where $x + y + z$ is largest — at Q. So the top of the object is Q, and along $\overline{PQ}$ its height above the cut is

$$\sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}.$$

(Check with the formula: $\frac{3-1}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.)

Watch out. The value $\frac{\sqrt{3}}{3}$, choice (A), is the height of the *discarded* tetrahedron. The object on the table is the rest of the cube, so you must subtract it from the full diagonal $\sqrt{3}$.

Answer: (D) $\dfrac{2\sqrt{3}}{3}$

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22 Counting & Probability

Source: 2025 AMC 10A #24

> Call a positive integer fair if no digit is used more than once, it has no 0s, and no digit is adjacent to two greater digits. For example, 23, 196, and 12463 are fair, but 1546, 320, and 34321 are not fair. How many fair positive integers are there?
>
> **(A)** 511 **(B)** 2,584 **(C)** 9,841 **(D)** 17,711 **(E)** 19,682

Key idea. "No digit is adjacent to two greater digits" says there is no dip (a digit smaller than both neighbors) in the middle of the number. Look at the examples: 12463 climbs to its largest digit and then falls. That suggests every fair number rises to its maximum and then falls, which turns the problem into choosing a side for each digit.

Step 1: fair means "up, then down". Only an interior digit has two neighbors, so fair means no interior digit is smaller than both of its neighbors. Let m be the largest digit. Suppose some step to the left of m goes down, say $d_i > d_{i+1}$. Then look at the stretch from d_i to m: its smallest digit is neither d_i (since d_{i+1} is smaller) nor m (the largest). So that smallest digit is interior and smaller than both neighbors — not fair. Hence the digits strictly increase up to m, and by the same argument they strictly decrease after it. Conversely, such a number has no dip, so it is fair.

Step 2: count for a fixed digit set. Choose a set of k distinct digits from 1–9. Its largest digit m is the peak. Each of the other $k-1$ digits goes either to the left of m or to the right. Once the sides are chosen, the left part must be written in increasing order and the right part in decreasing order. So this set gives 2^{k-1} fair numbers.

Step 3: add up. The total is

$$\sum_{k=1}^{9} \binom{9}{k} 2^{k-1} = \frac{1}{2} \sum_{k=1}^{9} \binom{9}{k} 2^{k} = \frac{1}{2}\left((1+2)^9 - 1\right) = \frac{19683 - 1}{2} = 9841.$$

(The middle step is the binomial theorem: $\sum_{k=0}^{9} \binom{9}{k} 2^k = 3^9$, minus the $k = 0$ term 1.) A computer search over all strings of distinct nonzero digits also finds exactly 9841.

Watch out. Letting every digit, the largest one included, choose a side gives $\sum \binom{9}{k} 2^k = 3^9 - 1 = 19682$, choice (E). The peak has no choice to make, so each count must be halved.

Answer: (C) 9,841

23 Geometry

Source: 2021 AMC 10A #24

The interior of a quadrilateral is bounded by the graphs of $(x+ay)^2 = 4a^2$ and $(ax-y)^2 = a^2$, where a is a positive real number. What is the area of this region in terms of a, valid for all $a > 0$?

(A) $\dfrac{8a^2}{(a+1)^2}$ **(B)** $\dfrac{4a}{a+1}$ **(C)** $\dfrac{8a}{a+1}$ **(D)** $\dfrac{8a^2}{a^2+1}$ **(E)** $\dfrac{8a}{a^2+1}$

Key idea. An equation of the form $(\text{linear})^2 = k^2$ is really two parallel lines, linear $= \pm k$. Two pairs of parallel lines bound a parallelogram; here the pairs turn out to be perpendicular, so the region is a rectangle whose sides are the distances between the parallel lines.

Step 1: the four lines. $(x+ay)^2 = 4a^2$ means $x+ay = 2a$ or $x+ay = -2a$: two parallel lines of slope $-\frac{1}{a}$. Likewise $(ax-y)^2 = a^2$ means $ax-y = a$ or $ax-y = -a$: two parallel lines of slope a.

Step 2: a rectangle. The slopes multiply to $a \cdot \left(-\frac{1}{a}\right) = -1$, so each pair is perpendicular to the other. The region is a rectangle.

Step 3: the side lengths. The distance between parallel lines $Ax+By = C_1$ and $Ax+By = C_2$ is $|C_1 - C_2|/\sqrt{A^2+B^2}$. For the first pair ($A=1$, $B=a$, $C=\pm 2a$) and the second pair ($A=a$, $B=-1$, $C=\pm a$) this gives

$$\frac{4a}{\sqrt{a^2+1}} \quad \text{and} \quad \frac{2a}{\sqrt{a^2+1}}.$$

Step 4: the area.

$$\frac{4a}{\sqrt{a^2+1}} \cdot \frac{2a}{\sqrt{a^2+1}} = \frac{8a^2}{a^2+1}.$$

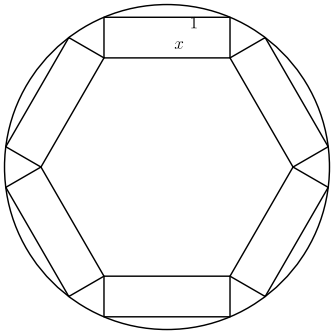
Watch out. Testing only $a = 1$: the lines $x+y = \pm 2$ and $x-y = \pm 1$ bound a rectangle of area 4, but choices (C), (D), and (E) all equal 4 at $a = 1$. That is why the problem says "valid for all $a > 0$". Try $a = 2$ as well: the true area is $\frac{32}{5}$, and only (D) gives that.

Answer: (D) $\dfrac{8a^2}{a^2+1}$

24 Geometry

Source: 2008 AMC 10A #25

A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being end points of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x?



(A) $2\sqrt{5}-\sqrt{3}$ **(B)** 3 **(C)** $\dfrac{3\sqrt{7}-\sqrt{3}}{2}$ **(D)** $2\sqrt{3}$ **(E)** $\dfrac{5+2\sqrt{3}}{2}$

Key idea. The six inner sides form a regular hexagon, whose geometry we know completely. Put the table's center at the origin, write the coordinates of one outer corner in terms of x, and use the fact that it lies on the circle of radius 4.

Step 1: the inner hexagon. Each mat's outer side is a chord of length x of the table, so every mat is a rotated copy of every other about the center O; in particular all inner corners are the same distance from O. The inner sides (length x) meet end to end at the six touching points, which therefore lie on one circle about O and are joined by six equal chords. Equal chords cut equal arcs, so the touching points are equally spaced: the inner sides form a regular hexagon of side x centered at O. A regular hexagon of side x is made of six equilateral triangles, so the distance from O to each side (the apothem) is $\frac{\sqrt{3}}{2}x$.

Step 2: an outer corner. Look at the top mat and put $O=(0,0)$. Its inner side is horizontal, from $\left(-\frac{x}{2},\frac{\sqrt{3}}{2}x\right)$ to $\left(\frac{x}{2},\frac{\sqrt{3}}{2}x\right)$. The mat has width 1, so its outer corners are 1 higher:

$$Q=\left(\frac{x}{2},\ \frac{\sqrt{3}}{2}x+1\right).$$

Step 3: Q is on the edge. So $OQ=4$:

$$\frac{x^2}{4}+\left(\frac{\sqrt{3}}{2}x+1\right)^2=16.$$

Expanding, $\frac{x^2}{4}+\frac{3x^2}{4}+\sqrt{3}\,x+1=16$, so

$$x^2+\sqrt{3}\,x-15=0.$$

Step 4: solve. By the quadratic formula,

$$x = \frac{-\sqrt{3} \pm \sqrt{3 + 60}}{2} = \frac{-\sqrt{3} \pm 3\sqrt{7}}{2}.$$

Length is positive, so $x = \dfrac{3\sqrt{7} - \sqrt{3}}{2} \approx 3.10$.

Watch out. The side of length x on the edge is *not* a side of a regular hexagon inscribed in the table — that would make $x = 4$, which is not a choice. The regular hexagon is the inner one, and the outer corners sit a width of 1 beyond it.

Answer: (C) $\dfrac{3\sqrt{7} - \sqrt{3}}{2}$

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25 Number Theory

Source: 2017 AMC 10A #25

How many integers between 100 and 999, inclusive, have the property that some permutation of its digits is a multiple of 11 between 100 and 999? For example, both 121 and 211 have this property.

(A) 226 **(B)** 243 **(C)** 270 **(D)** 469 **(E)** 486

Key idea. Whether a number qualifies depends only on its *digits*, not their order. So find every digit-set that can be arranged into a multiple of 11, then count how many three-digit numbers each digit-set makes. The tool is the divisibility rule: $\overline{ABC}$ is a multiple of 11 exactly when $A - B + C$ is.

Step 1: a cleaner form of the rule. Write $A - B + C = (A + B + C) - 2B$. So for a fixed set of digits, divisibility by 11 depends only on *which digit is in the middle*. Swapping the outer digits never matters, and with three different digits at most one choice of middle digit can work: if B and B' both worked, then 11 would divide $2(B - B')$, forcing $B = B'$.

There are 81 multiples of 11 from 110 to 990 (that is $11 \cdot 10$ up to $11 \cdot 90$).

Step 2: all three digits equal. Then $A - A + A = A$, a digit from 1 to 9 — never a multiple of 11. Nothing here.

Step 3: exactly two digits equal (x, x, y).

- y in the middle ($\overline{xyx}$): need $11 \mid 2x - y$. This gives $121, 242, 363, 484, 616, 737, 858, 979$. These 8 digit-sets have no 0, so each makes 3 numbers: $8 \cdot 3 = 24$.
- x in the middle ($\overline{xxy}$ or $\overline{yxx}$): the alternating sum is y, so $y = 0$. This gives $110, 220, \ldots, 990$. Each set $\{x, x, 0\}$ makes only $\overline{xx0}$ and $\overline{x0x}$ (0 cannot lead): $9 \cdot 2 = 18$.

That uses $8 + 9 = 17$ of the 81 multiples, leaving 64 with three different digits.

Step 4: three different digits, one of them 0. The 0 cannot be first. If it is last, $\overline{AB0}$ needs $A = B$, not allowed. So 0 is in the middle and $A + C = 11$: $209, 308, 407, 506, 605, 704, 803, 902$. These pair up into

4 digit-sets, $\{0,2,9\}, \{0,3,8\}, \{0,4,7\}, \{0,5,6\}$. Each makes $2 \cdot 2 = 4$ numbers (choose a nonzero first digit, then order the rest): $4 \cdot 4 = 16$.

Step 5: three different nonzero digits. The remaining $64 - 8 = 56$ multiples. By Step 1, each digit-set contains exactly two multiples, $\overline{ABC}$ and $\overline{CBA}$, so there are $56 \div 2 = 28$ digit-sets. Each makes $3! = 6$ numbers: $28 \cdot 6 = 168$.

Step 6: total. $24 + 18 + 16 + 168 = 226$.

Watch out. Multiplying all 81 multiples by $3! = 6$ gives 486, choice (E). That double-counts every pair $\overline{ABC}$, $\overline{CBA}$, and also counts arrangements with a leading 0 or repeated digits as different numbers.

Answer: (A) 226