AMC 10

PRACTICE EXAMINATION

SET 9

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 25 problems are taken from past AMC 10 contests (2000–2025) and appear in their original wording. Advanced level: each problem sits two or three places earlier than it did in its original contest, and problems 21–25 come from original positions 23–25. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**D**	2008 AMC 10B #3	Algebra
2	**D**	2012 AMC 10B #5	Algebra
3	**B**	2006 AMC 10A #6	Algebra
4	**C**	2010 AMC 10A #6	Algebra
5	**D**	2018 AMC 10B #7	Geometry
6	**B**	2019 AMC 10A #9	Number Theory
7	**A**	2011 AMC 10A #9	Geometry
8	**A**	2019 AMC 10B #11	Algebra
9	**D**	2020 AMC 10B #12	Number Theory
10	**C**	2014 AMC 10A #12	Geometry
11	**E**	2009 AMC 10A #13	Algebra
12	**B**	2023 AMC 10A #14	Counting & Probability
13	**B**	2010 AMC 10B #16	Geometry
14	**D**	2025 AMC 10A #16	Counting & Probability
15	**C**	2021 Fall AMC 10A #18	Counting & Probability
16	**A**	2021 Fall AMC 10B #19	Number Theory
17	**A**	2025 AMC 10B #20	Geometry
18	**C**	2024 AMC 10A #21	Algebra
19	**D**	2009 AMC 10B #21	Number Theory
20	**A**	2011 AMC 10B #22	Geometry
21	**B**	2017 AMC 10A #23	Counting & Probability
22	**D**	2018 AMC 10A #24	Geometry
23	**E**	2013 AMC 10A #24	Counting & Probability
24	**A**	2022 AMC 10B #25	Number Theory
25	**D**	2012 AMC 10A #25	Counting & Probability

1 Algebra

Source: 2008 AMC 10B #3

Assume that x is a positive real number. Which is equivalent to

$$\sqrt[3]{x\sqrt{x}}?$$

(A) $x^{\frac{1}{6}}$ **(B)** $x^{\frac{1}{4}}$ **(C)** $x^{\frac{3}{8}}$ **(D)** $x^{\frac{1}{2}}$ **(E)** x

Key idea. Rewrite every root as a fractional exponent. Then multiplying powers means *adding* exponents, and taking a root means *dividing* the exponent.

Step 1. Since $\sqrt{x} = x^{\frac{1}{2}}$, the expression under the cube root is

$$x\sqrt{x} = x^1 \cdot x^{\frac{1}{2}} = x^{1+\frac{1}{2}} = x^{\frac{3}{2}}.$$

Step 2. A cube root multiplies the exponent by $\frac{1}{3}$:

$$\sqrt[3]{x^{\frac{3}{2}}} = x^{\frac{3}{2}\cdot\frac{1}{3}} = x^{\frac{1}{2}}.$$

Check with $x = 64$: $64 \cdot \sqrt{64} = 64 \cdot 8 = 512$, and $\sqrt[3]{512} = 8 = \sqrt{64}$. ✓

Watch out. When multiplying $x \cdot x^{\frac{1}{2}}$ the exponents are *added*, not multiplied. Multiplying them gives $x^{\frac{1}{2}}$ under the cube root and then $x^{\frac{1}{6}}$, choice (A).

Answer: (D) $x^{\frac{1}{2}}$

2 Algebra

Source: 2012 AMC 10B #5

Anna enjoys dinner at a restaurant in Washington, D.C., where the sales tax on meals is 10%. She leaves a 15% tip on the price of her meal before the sales tax is added, and the tax is calculated on the pre-tip amount. She spends a total of \$27.50 for dinner. What is the cost of her dinner without tax or tip?

(A) \$18 **(B)** \$20 **(C)** \$21 **(D)** \$22 **(E)** \$24

Key idea. Both the tax and the tip are percentages of the *same* amount — the price of the meal — so they simply add up to one combined percentage.

Let the price of the meal be x dollars. The tax is 10% of x, which is $0.10x$, and the tip is 15% of x, which is $0.15x$. So the total she pays is

$$x + 0.10x + 0.15x = 1.25x.$$

Setting $1.25x = 27.50$ gives $x = \dfrac{27.50}{1.25} = 22$.

Check: the tax is \$2.20 and the tip is \$3.30, and $22 + 2.20 + 3.30 = 27.50$.

Watch out. Do not take 25% *off* \$27.50. The percentages are of the meal price, not of the total, and $0.75 \times 27.50 = 20.625$ is not even a whole number of cents — and not one of the choices.

Another way. Test the choices: $1.25 \times 22 = 27.50$ exactly, while $1.25 \times 20 = 25$ and $1.25 \times 24 = 30$.

Answer: (D) \$22

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3 Algebra

Source: 2006 AMC 10A #6

What non-zero real value for x satisfies $(7x)^{14} = (14x)^7$?

(A) $\frac{1}{7}$ **(B)** $\frac{2}{7}$ **(C)** 1 **(D)** 7 **(E)** 14

Key idea. The exponent 14 is exactly twice 7, so write the left side as a seventh power and compare the bases.

Since $(7x)^{14} = \left((7x)^2\right)^7$, the equation says $\left((7x)^2\right)^7 = (14x)^7$. Two real numbers with equal seventh powers are equal (an odd power never sends two different reals to the same value), so

$$(7x)^2 = 14x, \qquad \text{that is,} \qquad 49x^2 = 14x.$$

We are told $x \neq 0$, so we may divide both sides by $7x$: $7x = 2$, giving $x = \frac{2}{7}$.

Check: $7x = 2$ and $14x = 4$, and indeed $2^{14} = 16384 = 4^7$.

Watch out. Expanding $(7x)^2$ as $7x^2$ instead of $49x^2$ gives $7x^2 = 14x$, so $x = 2$ — not a choice. Square the 7 too.

Answer: (B) $\frac{2}{7}$

4 Algebra

Source: 2010 AMC 10A #6

For positive numbers x and y the operation $\spadesuit(x, y)$ is defined as

$$\spadesuit(x, y) = x - \frac{1}{y}$$

What is $\spadesuit(2, \spadesuit(2, 2))$?

(A) $\frac{2}{3}$ **(B)** 1 **(C)** $\frac{4}{3}$ **(D)** $\frac{5}{3}$ **(E)** 2

Key idea. Work from the inside out: evaluate the inner $\spadesuit(2, 2)$ first, then feed that number in as the new y.

Step 1. The inner operation has $x = 2$ and $y = 2$:

$$\spadesuit(2, 2) = 2 - \frac{1}{2} = \frac{3}{2}.$$

Step 2. Now the outer operation has $x = 2$ and $y = \frac{3}{2}$. The reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$, so

$$\spadesuit\left(2, \tfrac{3}{2}\right) = 2 - \frac{1}{3/2} = 2 - \frac{2}{3} = \frac{6}{3} - \frac{2}{3} = \frac{4}{3}.$$

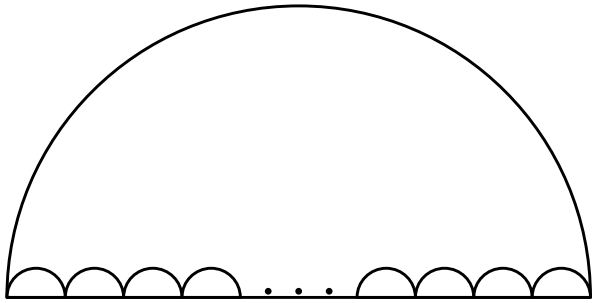
Watch out. Order matters: the inner result goes into the *second* slot. Putting it in the first slot instead computes $\spadesuit\left(\frac{3}{2}, 2\right) = \frac{3}{2} - \frac{1}{2} = 1$, which is choice (B).

Answer: (C) $\frac{4}{3}$

5 Geometry

Source: 2018 AMC 10B #7

In the figure below, N congruent semicircles are drawn along a diameter of a large semicircle, with their diameters covering the diameter of the large semicircle with no overlap. Let A be the combined area of the small semicircles and B be the area of the region inside the large semicircle but outside the small semicircles. The ratio $A : B$ is $1 : 18$. What is N?



(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 36

Key idea. Give the small radius a name, r. Because N small diameters line up exactly along the big diameter, the big radius is N times as long — so every area can be written in terms of r and N.

Step 1. The big diameter is N small diameters: $2R = N \cdot 2r$, so $R = Nr$.

Step 2. Each small semicircle has area $\frac{1}{2}\pi r^2$, so

$$A = N \cdot \tfrac{1}{2}\pi r^2.$$

The large semicircle has area $\frac{1}{2}\pi(Nr)^2 = N^2 \cdot \frac{1}{2}\pi r^2$. Region B is what is left over:

$$B = N^2 \cdot \tfrac{1}{2}\pi r^2 - N \cdot \tfrac{1}{2}\pi r^2 = (N^2 - N) \cdot \tfrac{1}{2}\pi r^2.$$

Step 3. The common factor $\frac{1}{2}\pi r^2$ cancels in the ratio:

$$A : B = N : (N^2 - N) = N : N(N-1) = 1 : (N-1).$$

Setting $1 : (N-1) = 1 : 18$ gives $N - 1 = 18$, so $N = 19$.

Watch out. B is only the part *outside* the small semicircles. If you compare A with the whole large semicircle instead, you get $A : (A + B) = 1 : N$, hence $N = 18$ — choice (C).

Answer: (D) 19

6 Number Theory

Source: 2019 AMC 10A #9

What is the greatest three-digit positive integer n for which the sum of the first n positive integers is not a divisor of the product of the first n positive integers?

(A) 995 **(B)** 996 **(C)** 997 **(D)** 998 **(E)** 999

Key idea. Write the sum as $\frac{n(n+1)}{2}$ and the product as $n!$. After cancelling the common factor n, everything depends on whether $n+1$ can be built from the numbers $1, 2, \ldots, n-1$ — which, for $n \geq 2$, fails exactly when $n+1$ is a prime.

Step 1. The sum is $\dfrac{n(n+1)}{2}$ and the product is $n! = n \cdot (n-1)!$. Their quotient is

$$\frac{n!}{n(n+1)/2} = \frac{2\,(n-1)!}{n+1}.$$

So the sum divides the product exactly when $n+1$ divides $2 \cdot (n-1)!$.

Step 2. Check from the top, using $(n-1)! = 1 \cdot 2 \cdots (n-1)$.

- $n = 999$: $n+1 = 1000 = 8 \cdot 125$. The factors 8 and 125 both appear separately in 998!, so 1000 divides it. Divides.
- $n = 998$: $n+1 = 999 = 27 \cdot 37$. The factors 27 and 37 both appear in 997!. Divides.
- $n = 997$: $n+1 = 998 = 2 \cdot 499$. The factors 2 and 499 both appear in 996!. Divides.
- $n = 996$: $n+1 = 997$, which is prime. Every factor of $2 \cdot 995!$ is less than 997, so the prime 997 does not divide it. Does *not* divide.

So the greatest such n is 996.

Watch out. Seeing that 997 is prime and answering $n = 997$ (C). The prime has to be $n+1$, not n: for $n = 997$ the sum does divide the product.

Answer: (B) 996

7 Geometry

Source: 2011 AMC 10A #9

A rectangular region is bounded by the graphs of the equations $y = a$, $y = -b$, $x = -c$, and $x = d$, where a, b, c, and d are all positive numbers. Which of the following represents the area of this region?

(A) $ac + ad + bc + bd$ **(B)** $ac - ad + bc - bd$ **(C)** $ac + ad - bc - bd$

(D) $-ac - ad + bc + bd$ **(E)** $ac - ad - bc + bd$

Key idea. A side length is the distance between two parallel lines: the larger coordinate minus the smaller one. Since a, b, c, d are positive, $-b$ is below the x-axis and $-c$ is to the left of the y-axis.

Height. The horizontal sides are $y = a$ (above the x-axis) and $y = -b$ (below it). The height is $a - (-b) = a + b$.

Width. The vertical sides are $x = d$ (right of the y-axis) and $x = -c$ (left of it). The width is $d - (-c) = c + d$.

Area. Multiply and expand:

$$(a + b)(c + d) = ac + ad + bc + bd.$$

Watch out. Using $a-b$ and $c-d$ as the side lengths (forgetting that subtracting a negative coordinate adds) gives $(a - b)(c - d) = ac - ad - bc + bd$, which is choice (E).

Answer: (A) $ac + ad + bc + bd$

8 Algebra

Source: 2019 AMC 10B #11

Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is $9 : 1$, and the ratio of blue to green marbles in Jar 2 is $8 : 1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

(A) 5 **(B)** 10 **(C)** 25 **(D)** 45 **(E)** 50

Key idea. A ratio $9 : 1$ means the jar splits into 10 equal parts, so name the size of one part. Here one "part" is exactly the number of green marbles, so let the green counts be the unknowns.

Step 1. Let Jar 1 have x green marbles and Jar 2 have y green marbles. Then Jar 1 has $9x$ blue marbles, $10x$ marbles in total; Jar 2 has $8y$ blue marbles, $9y$ in total.

Step 2. The two conditions are

$$10x = 9y \qquad \text{and} \qquad x + y = 95.$$

From the first, $y = \frac{10}{9}x$, so $x + \frac{10}{9}x = \frac{19}{9}x = 95$, giving $x = 45$ and $y = 50$. Check: both jars hold 450 marbles.

Step 3. Jar 1 has $9 \cdot 45 = 405$ blue marbles and Jar 2 has $8 \cdot 50 = 400$. The difference is $405 - 400 = 5$.

Answer: (A) 5

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9 Number Theory

Source: 2020 AMC 10B #12

> The decimal representation of
> $$\frac{1}{20^{20}}$$
> consists of a string of zeros after the decimal point, followed by a 9 and then several more digits. How many zeros are in that initial string of zeros after the decimal point?
>
> **(A)** 23 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27

Key idea. Dividing by a power of 10 just slides the decimal point. So rewrite $\frac{1}{20^{20}}$ as (some whole number) $\div$ (a power of 10): multiply top and bottom by 5^{20}.

Step 1. Since $20 \cdot 5 = 100$,
$$\frac{1}{20^{20}} = \frac{5^{20}}{20^{20} \cdot 5^{20}} = \frac{5^{20}}{100^{20}} = \frac{5^{20}}{10^{40}}.$$

Step 2. Size up 5^{20}. We have $5^{20} = \dfrac{10^{20}}{2^{20}}$ and $2^{20} = 1{,}048{,}576 \approx 1.05 \times 10^6$, so $5^{20} \approx \dfrac{10^{14}}{1.05} \approx 0.95 \times 10^{14}$. (Exactly, $5^{20} = 95{,}367{,}431{,}640{,}625$.) So 5^{20} has 14 digits and begins with 9.

Step 3. Dividing a 14-digit number by 10^{40} puts its last digit in the 40th place after the decimal point, so its first digit, the 9, lands in place $40 - 14 + 1 = 27$. The places before it, $40 - 14 = 26$ of them, are zeros:
$$\frac{1}{20^{20}} = 0.\underbrace{00\cdots0}_{26}95367431640625.$$

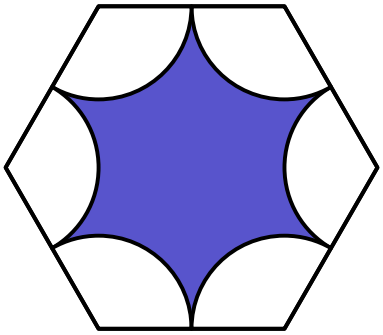
Watch out. $20^{20} = 2^{20} \cdot 10^{20}$ has $7 + 20 = 27$ digits, and it is tempting to say "27 digits, so 27 zeros," which is (E). But 1 divided by a 27-digit number that starts with 1 is between 10^{-27} and 10^{-26}, so it has only 26 zeros before its first nonzero digit.

Answer: (D) 26

10 Geometry

Source: 2014 AMC 10A #12

A regular hexagon has side length 6. Congruent arcs with radius 3 are drawn with the center at each of the vertices, creating circular sectors as shown. The region inside the hexagon but outside the sectors is shaded as shown. What is the area of the shaded region?



(A) $27\sqrt{3}-9\pi$ **(B)** $27\sqrt{3}-6\pi$ **(C)** $54\sqrt{3}-18\pi$ **(D)** $54\sqrt{3}-12\pi$ **(E)** $108\sqrt{3}-9\pi$

Key idea. Shaded area = hexagon − sectors. The only thing to get right is the angle of each sector: it is the interior angle of the hexagon at that vertex.

Step 1. Lines from the center to the six vertices split a regular hexagon of side 6 into six equilateral triangles of side 6. Each has area $\frac{\sqrt{3}}{4}\cdot 6^2 = 9\sqrt{3}$, so the hexagon has area

$$6\cdot 9\sqrt{3} = 54\sqrt{3}.$$

Step 2. Each sector is centered at a vertex and bounded by the two sides that meet there, so its angle is the interior angle of a regular hexagon, 120°. Its radius 3 is half a side, so arcs from neighboring vertices meet exactly at the midpoint of the side between them and the sectors do not overlap.

Step 3. A 120° sector is $\frac{120}{360}=\frac{1}{3}$ of a circle of area $\pi\cdot 3^2 = 9\pi$, so it has area 3π. The six sectors total $6\cdot 3\pi = 18\pi$ — the same as two full circles of radius 3.

Step 4. The shaded area is $54\sqrt{3}-18\pi \approx 93.5-56.5 = 37.0$, which is positive, as it should be.

Watch out. The triangles in Step 1 have 60° angles, but the sectors sit at the hexagon's *corners*, where two triangles meet, so each sector is 120°. Using 60° gives sectors totalling 9π and the answer $54\sqrt{3}-9\pi$, which is not among the choices.

Answer: (C) $54\sqrt{3}-18\pi$

11 Algebra

Source: 2009 AMC 10A #13

Suppose that $P = 2^m$ and $Q = 3^n$. Which of the following is equal to 12^{mn} for every pair of integers (m, n)?

(A) P^2Q **(B)** P^nQ^m **(C)** P^nQ^{2m} **(D)** $P^{2m}Q^n$ **(E)** $P^{2n}Q^m$

Key idea. Every choice is built from $P = 2^m$ and $Q = 3^n$, powers of the primes 2 and 3 — exactly the primes in 12. That says: break 12 into primes, $12 = 2^2 \cdot 3$, and then regroup the exponents so that the powers of 2 look like $(2^m)^{\text{something}}$ and the powers of 3 look like $(3^n)^{\text{something}}$.

Step 1. Since $12 = 2^2 \cdot 3$,

$$12^{mn} = \left(2^2 \cdot 3\right)^{mn} = 2^{2mn} \cdot 3^{mn}.$$

Step 2. Group the powers of 2 around m: $2^{2mn} = (2^m)^{2n} = P^{2n}$.

Step 3. Group the powers of 3 around n: $3^{mn} = (3^n)^m = Q^m$.

So $12^{mn} = P^{2n}Q^m$ for every pair of integers (m, n).

Watch out. Testing with $m = n = 1$ does not settle it: then $12^{mn} = 12$, and (A) gives $4 \cdot 3 = 12$, (D) gives $4 \cdot 3 = 12$, and (E) gives $4 \cdot 3 = 12$. Pick values with $m \neq n$, such as $m = 1$, $n = 2$: now $12^{mn} = 144$, while (A) gives 36, (D) gives 324, and only (E) gives $16 \cdot 9 = 144$.

Answer: (E) $P^{2n}Q^m$

12 Counting & Probability

Source: 2023 AMC 10A #14

A number is chosen at random from among the first 100 positive integers, and a positive integer divisor of that number is then chosen at random. What is the probability that the chosen divisor is divisible by 11?

(A) $\frac{4}{100}$ **(B)** $\frac{9}{200}$ **(C)** $\frac{1}{20}$ **(D)** $\frac{11}{200}$ **(E)** $\frac{3}{50}$

Key idea. A divisor can be a multiple of 11 only if the number itself is. So first ask how likely the number is a multiple of 11, then — given that it is — what fraction of its divisors are multiples of 11. Because $100 < 121 = 11^2$, the second fraction turns out to be the same for every such number.

Step 1. If 11 divides a divisor of n, then 11 divides n. The multiples of 11 up to 100 are $11, 22, \ldots, 99$ — that is 9 numbers out of 100. For any other n the probability of success is 0.

Step 2. Take $n = 11m$ with $1 \leq m \leq 9$. Since $m < 11$, the prime 11 does not divide m, so every divisor of $11m$ is either a divisor d of m or $11d$. These two lists have the same length, and exactly the second list consists of multiples of 11. So exactly half of the divisors of n are divisible by 11. For example, 22 has divisors $1, 2, 11, 22$, and two of the four are multiples of 11.

Step 3. Multiply the two stages:

$$\frac{9}{100} \cdot \frac{1}{2} = \frac{9}{200}.$$

Watch out. Stopping after Step 1 gives $\frac{9}{100}$ — the chance the *number* is a multiple of 11. That forgets the second random choice, and $\frac{9}{100}$ is not among the choices.

Answer: (B) $\frac{9}{200}$

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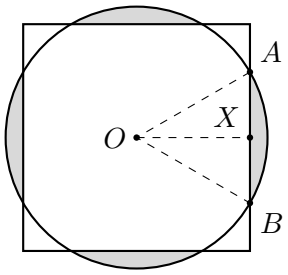
13 Geometry

Source: 2010 AMC 10B #16

A square of side length 1 and a circle of radius $\frac{\sqrt{3}}{3}$ share the same center. What is the area inside the circle, but outside the square?

(A) $\frac{\pi}{3} - 1$ **(B)** $\frac{2\pi}{9} - \frac{\sqrt{3}}{3}$ **(C)** $\frac{\pi}{18}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{2\pi}{9}$

Key idea. First decide how the two shapes overlap. The radius $\frac{\sqrt{3}}{3} \approx 0.577$ is more than half the side (0.5) but less than half the diagonal ($\frac{\sqrt{2}}{2} \approx 0.707$), so the circle pokes out through the middle of each side while the corners of the square stick out of the circle. The answer is four equal circular caps, and each cap is "sector minus triangle".



Step 1. Let the circle cross the right side of the square at A and B, and let X be the midpoint of that side, so $OX = \frac{1}{2}$. In right triangle OXA, $AX^2 = OA^2 - OX^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, so $AX = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$.

Step 2. Then $AB = 2 \cdot \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$, which equals the radius. So $\triangle OAB$ is equilateral and $\angle AOB = 60°$. (Equivalently, $\triangle OXA$ is a 30-60-90 triangle.)

Step 3. The sector AOB is $\frac{60}{360} = \frac{1}{6}$ of the circle: its area is $\frac{1}{6} \cdot \pi \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{1}{6} \cdot \frac{\pi}{3} = \frac{\pi}{18}$. The equilateral triangle OAB has area $\frac{\sqrt{3}}{4} \cdot \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{\sqrt{3}}{4} \cdot \frac{1}{3} = \frac{\sqrt{3}}{12}$.

Step 4. One cap (the shaded part to the right of $\overline{AB}$) has area $\frac{\pi}{18} - \frac{\sqrt{3}}{12}$. There are four, one at each

side, and they do not overlap because the corners are outside the circle. The total is

$$4\left(\frac{\pi}{18}-\frac{\sqrt{3}}{12}\right)=\frac{2\pi}{9}-\frac{\sqrt{3}}{3}.$$

Watch out. Subtracting areas, $\pi\left(\frac{\sqrt{3}}{3}\right)^2-1=\frac{\pi}{3}-1$ — choice (A) — assumes the square lies entirely inside the circle. It does not: its corners are 0.707 from the center. Adding four sectors and forgetting to remove the triangles gives $4\cdot\frac{\pi}{18}=\frac{2\pi}{9}$ — choice (E).

Answer: (B) $\frac{2\pi}{9}-\frac{\sqrt{3}}{3}$

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14 Counting & Probability

Source: 2025 AMC 10A #16

> There are three jars. Each of three coins is placed in one of the three jars, chosen at random and independently of the placements of the other coins. What is the expected number of coins in a jar with the most coins?
>
> **(A)** $\frac{4}{3}$ **(B)** $\frac{39}{27}$ **(C)** $\frac{5}{3}$ **(D)** $\frac{17}{9}$ **(E)** 2

Key idea. With only $3^3=27$ equally likely placements, sort them by the *shape* of the split — 3-0-0, 2-1-0, or 1-1-1 — and the largest jar is fixed in each shape.

Step 1. Each coin has 3 choices, so there are $3\cdot3\cdot3=27$ equally likely placements.

Step 2. Count each shape.

- All three coins in one jar (3-0-0): choose the jar, 3 ways. The fullest jar has 3 coins.
- One coin in each jar (1-1-1): the coins go to different jars in $3!=6$ ways. The fullest jar has 1 coin.
- The rest (2-1-0): $27-3-6=18$ ways. (Directly: 3 choices of the jar with two coins, 2 of the jar with one, and 3 of which coin is alone: $3\cdot2\cdot3=18$.) The fullest jar has 2 coins.

Step 3. The expected value is

$$\frac{3\cdot3+18\cdot2+6\cdot1}{27}=\frac{9+36+6}{27}=\frac{51}{27}=\frac{17}{9}.$$

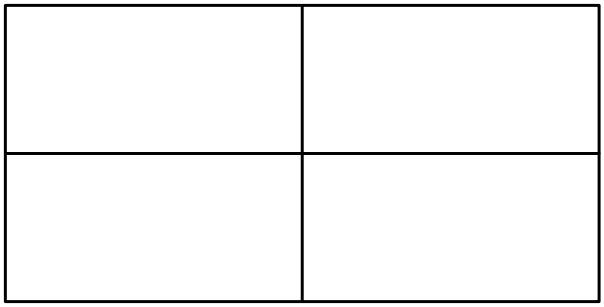
Watch out. Do not treat the three shapes as equally likely: they occur 3, 18 and 6 times out of 27. Averaging $3,2,1$ equally gives 2, choice (E).

Answer: (D) $\frac{17}{9}$

15 Counting & Probability

Source: 2021 Fall AMC 10A #18

A farmer's rectangular field is partitioned into 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?



(A) 12 **(B)** 64 **(C)** 84 **(D)** 90 **(E)** 144

Key idea. Each crop has exactly one "enemy" (corn–wheat, soybeans–potatoes). The two diagonal sections do not share a border, and both of them border each of the other two sections. So split into cases on whether the diagonal pair is the same crop.

Setup. Label the sections top-left, top-right, bottom-left, bottom-right. Top-right and bottom-left are diagonal (no shared border); each of the other two sections borders *both* of them. Same crop next to itself is allowed.

Case 1: top-right and bottom-left have the same crop. There are 4 choices for that crop. Each remaining section borders only this one crop, so it must avoid just its enemy: 3 choices each. That gives $4 \cdot 3 \cdot 3 = 36$ ways.

Case 2: top-right and bottom-left have different crops. There are $4 \cdot 3 = 12$ ordered choices. Now each remaining section must avoid the enemy of each of the two crops.

- If the two crops are enemies (e.g. corn and wheat), the forbidden crops are wheat and corn — just 2 crops, leaving 2 choices.
- If they are not enemies (e.g. corn and soybeans), the forbidden crops are wheat and potatoes — again 2 crops, leaving 2 choices.

Either way each remaining section has 2 choices, giving $12 \cdot 2 \cdot 2 = 48$ ways.

Total. The two cases do not overlap and cover everything, so there are $36 + 48 = 84$ ways.

Watch out. Choosing sections in order and saying "the last section borders two others, so 3 choices" gives $4 \cdot 3 \cdot 3 \cdot 3 = 108$, not a choice. The last section has 3 choices only when its two neighbours share a crop; that is why the cases are needed.

Answer: (C) 84

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16 Number Theory

Source: 2021 Fall AMC 10B #19

Let N be the positive integer $7777\ldots777$, a 313-digit number where each digit is a 7. Let $f(r)$ be the leading digit of the rth root of N. What is

$$f(2) + f(3) + f(4) + f(5) + f(6)?$$

(A) 8 **(B)** 9 **(C)** 11 **(D)** 22 **(E)** 29

Key idea. The leading digit of a number does not change when you multiply by a power of 10. So pull out of N a power of 10 whose rth root is a whole power of 10, and only a small "front part" is left to take the root of. The 313 digits matter only through the remainder of 312 when divided by r.

Step 1. $N = 7.77\ldots \times 10^{312}$, so $7 \cdot 10^{312} < N < 8 \cdot 10^{312}$.

Step 2 ($r = 2, 3, 4, 6$). Each of these divides 312 ($312 = 2 \cdot 156 = 3 \cdot 104 = 4 \cdot 78 = 6 \cdot 52$), so $\sqrt[r]{10^{312}}$ is a power of 10 and the rth root of N has the same leading digit as $\sqrt[r]{x}$ for some x with $7 < x < 8$.

- $r = 2$: $2^2 = 4 < 7$ and $8 < 9 = 3^2$, so $2 < \sqrt{x} < 3$ and $f(2) = 2$.
- $r = 3$: $1 < 7$ and $8 = 2^3$, so $1 < \sqrt[3]{x} < 2$ and $f(3) = 1$.
- $r = 4$: $8 < 16 = 2^4$, so $f(4) = 1$.
- $r = 6$: $8 < 64 = 2^6$, so $f(6) = 1$.

Step 3 ($r = 5$). Here $312 = 5 \cdot 62 + 2$, so keep two extra powers of 10 in the front part: $700 \cdot 10^{310} < N < 800 \cdot 10^{310}$, and $\sqrt[5]{10^{310}} = 10^{62}$. Since $3^5 = 243 < 700$ and $800 < 1024 = 4^5$, the fifth root of the front part is between 3 and 4, so $f(5) = 3$.

Step 4. The sum is $2 + 1 + 1 + 3 + 1 = 8$.

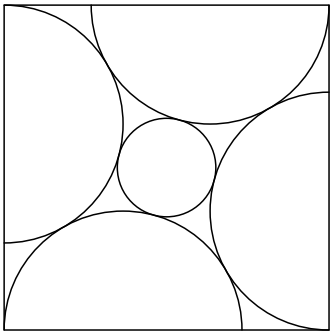
Watch out. Treating $r = 5$ like the others and taking the fifth root of $7.77\ldots$ gives $f(5) = 1$ and a sum of 6, which is not a choice. The front part must be chosen so that the power of 10 left over is a multiple of r.

Answer: (A) 8

17 Geometry

Source: 2025 AMC 10B #20

Four congruent semicircles are inscribed in a square of side length 1 so that their diameters are on the sides of the square, one endpoint of each diameter is at a vertex of the square, and adjacent semicircles are tangent to each other. A small circle centered at the center of the square is tangent to each of the four semicircles, as shown below.



The diameter of the small circle can be written as $(\sqrt{a}+b)(\sqrt{c}+d)$, where a, b, c, and d are integers. What is $a+b+c+d$?

(A) 3 **(B)** 5 **(C)** 8 **(D)** 9 **(E)** 11

Key idea. Every condition in the statement is a tangency, and two circles that touch from outside have their centers exactly (sum of radii) apart. Put the square on coordinates so every center has simple coordinates, and the tangencies become equations.

Step 1. Place the square with corners $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ and let each semicircle have radius r. As in the figure, the bottom semicircle has its diameter from $(0,0)$ to $(2r,0)$, so its center is $(r,0)$; the right one has its diameter from $(1,0)$ up to $(1,2r)$, so its center is $(1,r)$.

Step 2. These two are tangent, so their centers are $r+r=2r$ apart:

$$(1-r)^2+r^2=4r^2 \implies 1-2r+2r^2=4r^2 \implies 2r^2+2r-1=0.$$

The positive root is $r=\dfrac{-2+\sqrt{12}}{4}=\dfrac{\sqrt{3}-1}{2}\approx 0.366$.

Step 3. The small circle has center $\left(\frac{1}{2},\frac{1}{2}\right)$ and radius t. Its distance to $(r,0)$ uses $\frac{1}{2}-r=\frac{2-\sqrt{3}}{2}$:

$$\left(\frac{2-\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2=\frac{7-4\sqrt{3}}{4}+\frac{1}{4}=2-\sqrt{3}.$$

So the distance is $\sqrt{2-\sqrt{3}}$. This nested root simplifies, because

$$\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2=\frac{6+2-2\sqrt{12}}{4}=\frac{8-4\sqrt{3}}{4}=2-\sqrt{3},$$

and $\frac{\sqrt{6}-\sqrt{2}}{2}>0$, so $\sqrt{2-\sqrt{3}}=\frac{\sqrt{6}-\sqrt{2}}{2}$.

Step 4. Tangency gives $t + r = \frac{\sqrt{6}-\sqrt{2}}{2}$. The diameter is

$$2t = (\sqrt{6} - \sqrt{2}) - (\sqrt{3} - 1) = \sqrt{6} - \sqrt{3} - \sqrt{2} + 1 = (\sqrt{3} - 1)(\sqrt{2} - 1).$$

(Expand the right side to check.) By the square's 90° rotational symmetry this circle is tangent to all four semicircles. Numerically $2t \approx 0.303$, matching the picture.

Step 5. So $a = 3$, $b = -1$, $c = 2$, $d = -1$, and $a + b + c + d = 3 - 1 + 2 - 1 = 3$.

Watch out. b and d are negative. Adding $3 + 1 + 2 + 1 = 7$ ignores the signs, and 7 is not a choice.

Answer: (A) 3

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18 Algebra

Source: 2024 AMC 10A #21

The numbers, in order, of each row and the numbers, in order, of each column of a 5×5 array of integers form an arithmetic progression of length 5. The numbers in positions $(5,5)$, $(2,4)$, $(4,3)$, and $(3,1)$ are 0, 48, 16, and 12, respectively. What number is in position $(1,2)$?

$$\begin{bmatrix} \cdot & ? & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 48 & \cdot \\ 12 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 16 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

(A) 19 **(B)** 24 **(C)** 29 **(D)** 34 **(E)** 39

Key idea. "Arithmetic in every row and every column" is a strong condition: it forces the whole array into a formula with only four unknown constants. Four entries are given — exactly enough to find them. Measuring positions from the centre $(3,3)$ keeps the numbers small.

Step 1: the shape of the array. Write $a(i,j)$ for the entry in row i, column j, and put $u = i-3$, $v = j-3$, so $u, v \in \{-2, \ldots, 2\}$. Row i is arithmetic, so $a = p_i + q_i v$ for some numbers p_i (the middle entry of the row) and q_i (its common difference).

Column 3 is arithmetic, so its entries p_i change by a constant amount as i goes down: $p_i = P + Qu$. Column 4 is arithmetic too, so $p_i + q_i$ is linear in u, and subtracting, $q_i = R + Su$ is linear in u as well. Therefore

$$a = P + Qu + Rv + Suv.$$

Here P is the centre entry, Q is the step down column 3, R the step along row 3.

Step 2: use the four entries. With (u, v) for each given position:

$$\begin{aligned} (4,3) \to (1,0): &\quad P + Q = 16, \\ (3,1) \to (0,-2): &\quad P - 2R = 12, \\ (2,4) \to (-1,1): &\quad P - Q + R - S = 48, \\ (5,5) \to (2,2): &\quad P + 2Q + 2R + 4S = 0. \end{aligned}$$

Step 3: solve. From the first two, $P = 16 - Q$ and $2R = P - 12 = 4 - Q$. Put these into the last equation: $(16 - Q) + 2Q + (4 - Q) + 4S = 0$, so $20 + 4S = 0$ and $S = -5$.

Now the third equation: $(16 - Q) - Q + \frac{4-Q}{2} + 5 = 48$. Multiply by 2: $32 - 4Q + 4 - Q + 10 = 96$, that is, $46 - 5Q = 96$, so $-5Q = 50$ and $Q = -10$. Then $P = 26$ and $R = 7$.

Step 4: the answer. Position $(1, 2)$ has $(u, v) = (-2, -1)$:

$$a = 26 + (-10)(-2) + 7(-1) + (-5)(-2)(-1) = 26 + 20 - 7 - 10 = 29.$$

As a check, the full array has rows $12, 29, 46, 63, 80$; $12, 24, 36, 48, 60$; $12, 19, 26, 33, 40$; $12, 14, 16, 18, 20$; $12, 9, 6, 3, 0$. Every row and every column is arithmetic, all entries are integers, and the four given numbers are in place.

Answer: (C) 29

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19 Number Theory

Source: 2009 AMC 10B #21

> What is the remainder when
>
> $$3^0 + 3^1 + 3^2 + \cdots + 3^{2009}$$
>
> is divided by 8?
>
> **(A)** 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 6

Key idea. Powers of 3 repeat quickly modulo 8: $3^0 = 1$, $3^1 = 3$, $3^2 = 9 \equiv 1$, $3^3 = 27 \equiv 3$, … So the terms come in pairs with the same remainder pattern.

Step 1. Since $3^2 = 9 = 8 + 1$, multiplying by 9 does not change a remainder mod 8. Hence every even power of 3 leaves remainder 1 and every odd power leaves remainder 3.

Step 2. The exponents $0, 1, \ldots, 2009$ give 2010 terms, forming 1005 consecutive pairs $3^{2j} + 3^{2j+1}$. Each pair leaves remainder $1 + 3 = 4$.

Step 3. The total leaves the same remainder as $1005 \cdot 4 = 4020 = 8 \cdot 502 + 4$, so the remainder is 4.

Another way. The geometric-series formula gives $\dfrac{3^{2010} - 1}{2}$. Because of the division by 2, work mod 16: $3^{2010} = 9^{1005}$ and $9^2 = 81 \equiv 1 \pmod{16}$, so $9^{1005} \equiv 9 \pmod{16}$. Then $3^{2010} - 1 \equiv 8 \pmod{16}$, and halving gives remainder 4 mod 8.

Watch out. From $3^{2010} \equiv 1 \pmod 8$ it is tempting to say $\frac{3^{2010}-1}{2} \equiv \frac{0}{2} = 0$, choice (A). You cannot divide by 2 inside "mod 8" — that is why the alternate method uses mod 16.

Answer: (D) 4

20 Geometry

Source: 2011 AMC 10B #22

A pyramid has a square base with sides of length 1 and has lateral faces that are equilateral triangles. A cube is placed within the pyramid so that one face is on the base of the pyramid and its opposite face has all its edges on the lateral faces of the pyramid. What is the volume of this cube?

(A) $5\sqrt{2}-7$ **(B)** $7-4\sqrt{3}$ **(C)** $\frac{2\sqrt{2}}{27}$ **(D)** $\frac{\sqrt{2}}{9}$ **(E)** $\frac{\sqrt{3}}{9}$

Key idea. The top face of the cube sits at height x (the side length) and fits exactly inside the pyramid's horizontal cross-section at that height. So we need the pyramid's height, and how its square cross-sections shrink as we go up.

Step 1: the height. Each lateral edge is a side of an equilateral triangle with side 1, so it has length 1. The base diagonal is $\sqrt{2}$, so the apex is directly above the center, at horizontal distance $\frac{\sqrt{2}}{2}$ from each base corner. The height is

$$H=\sqrt{1^2-\left(\tfrac{\sqrt{2}}{2}\right)^2}=\sqrt{\tfrac{1}{2}}=\tfrac{\sqrt{2}}{2}.$$

Step 2: cross-sections. At height t the pyramid's cross-section is a square similar to the base, scaled by $\frac{H-t}{H}=1-\sqrt{2}\,t$. So its side is $1-\sqrt{2}\,t$.

Step 3: fit the cube. Let the cube have side x. Its top face is at height x, and "all its edges on the lateral faces" means that this face is exactly the cross-section square at height x. Hence

$$x=1-\sqrt{2}\,x \quad\Longrightarrow\quad x=\frac{1}{1+\sqrt{2}}=\sqrt{2}-1.$$

Step 4: the volume. $(\sqrt{2}-1)^2=3-2\sqrt{2}$, so

$$(\sqrt{2}-1)^3=(3-2\sqrt{2})(\sqrt{2}-1)=3\sqrt{2}-3-4+2\sqrt{2}=5\sqrt{2}-7.$$

Watch out. Using the slant height $\frac{\sqrt{3}}{2}$ of a lateral face as the pyramid's height. That gives $x=2\sqrt{3}-3$ and a volume of about 0.100, which matches none of the choices. The height must be measured straight up from the base, not along a face.

Answer: (A) $5\sqrt{2}-7$

21 Counting & Probability

Source: 2017 AMC 10A #23

How many triangles with positive area have all their vertices at points (i, j) in the coordinate plane, where i and j are integers between 1 and 5, inclusive?

(A) 2128 **(B)** 2148 **(C)** 2160 **(D)** 2200 **(E)** 2300

Key idea. Counting good triangles directly is hopeless, but counting *bad* ones is not: three points fail to make a triangle exactly when they are collinear. So count all triples, then subtract the collinear triples line by line.

Step 1. There are $5 \cdot 5 = 25$ points, so the number of ways to choose 3 of them is

$$\binom{25}{3} = \frac{25 \cdot 24 \cdot 23}{6} = 2300.$$

Step 2: which lines matter? A line through three grid points steps from one to the next by the same move (a, b), with a and b having no common factor. Two such steps cover $2a$ across and $2b$ up, and that must fit inside a 5×5 grid, so $|a| \le 2$ and $|b| \le 2$. The only possible directions are horizontal, vertical, slopes ± 1, and slopes ± 2, $\pm \frac{1}{2}$.

Step 3: count collinear triples. A line with k grid points contains $\binom{k}{3}$ collinear triples.

- Rows, columns, and the two long diagonals: $5 + 5 + 2 = 12$ lines with 5 points, giving $12 \cdot \binom{5}{3} = 12 \cdot 10 = 120$.
- Diagonals (slope ± 1) with 4 points, such as $(1, 2)$ to $(4, 5)$: 4 lines, giving $4 \cdot \binom{4}{3} = 16$.
- Diagonals with 3 points, such as $(1, 3)$ to $(3, 5)$: 4 lines, giving $4 \cdot 1 = 4$.
- Slope 2: a step of $(1, 2)$ twice rises 4, so y runs $1, 3, 5$ and x can start at 1, 2, or 3 — 3 lines of exactly 3 points. The same holds for slopes -2, $\frac{1}{2}$, $-\frac{1}{2}$: $4 \cdot 3 = 12$ lines, giving 12.

In total there are $120 + 16 + 4 + 12 = 152$ collinear triples.

Step 4. The number of triangles with positive area is

$$2300 - 152 = 2148.$$

Watch out. It is easy to check only rows, columns and 45° diagonals. Forgetting the slopes ± 2 and $\pm \frac{1}{2}$ gives $2300 - 120 - 16 - 4 = 2160$, choice (C). Forgetting collinear triples altogether gives 2300, choice (E).

Answer: (B) 2148

22 Geometry

Source: 2018 AMC 10A #24

Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

(A) 60 **(B)** 65 **(C)** 70 **(D)** 75 **(E)** 80

Key idea. Quadrilateral $FDBG$ is triangle ABG with the small corner triangle ADF cut off. The angle bisector tells us how much of $\triangle ABC$ is $\triangle ABG$, and the midsegment $\overline{DE} \parallel \overline{BC}$ makes $\triangle ADF$ a half-size copy of $\triangle ABG$.

Step 1. Area of $\triangle ABG$. By the Angle Bisector Theorem,

$$\frac{BG}{GC} = \frac{AB}{AC} = \frac{50}{10} = 5, \quad \text{so} \quad BG = \frac{5}{6}\,BC.$$

Triangles ABG and ABC have the same height from A to line BC, so their areas are in the ratio of their bases:

$$[ABG] = \frac{5}{6} \cdot 120 = 100.$$

Step 2. Area of $\triangle ADF$. D and E are midpoints, so $\overline{DE}$ is a midsegment and $\overline{DE} \parallel \overline{BC}$. Hence $\overline{DF} \parallel \overline{BG}$, and $\triangle ADF$ is similar to $\triangle ABG$ with ratio $\frac{AD}{AB} = \frac{1}{2}$ (in particular, F is the midpoint of $\overline{AG}$). Areas scale by the square of the ratio:

$$[ADF] = \left(\frac{1}{2}\right)^2 \cdot 100 = 25.$$

Step 3. Since D lies on $\overline{AB}$ and F lies on $\overline{AG}$, removing $\triangle ADF$ from $\triangle ABG$ leaves exactly quadrilateral $FDBG$:

$$[FDBG] = 100 - 25 = 75.$$

Notice that the lengths 50 and 10 were used only through their ratio 5, and we never needed the angle at A or the length BC.

Watch out. Similar figures with side ratio $\frac{1}{2}$ have area ratio $\frac{1}{4}$, not $\frac{1}{2}$. Taking $[ADF] = \frac{1}{2} \cdot 100 = 50$ gives $100 - 50 = 50$, which is not a choice.

Answer: (D) 75

23 Counting & Probability

Source: 2013 AMC 10A #24

> Central High School is competing against Northern High School in a backgammon match. Each school has three players, and the contest rules require that each player play two games against each of the other school's players. The match takes place in six rounds, with three games played simultaneously in each round. In how many different ways can the match be scheduled?
>
> **(A)** 540 **(B)** 600 **(C)** 720 **(D)** 810 **(E)** 900

Key idea. Each round pairs every Central player with a different Northern player, so a round is one of only $3! = 6$ possible *pairings*. A schedule is an ordered list of six pairings, so first decide *which* pairings (with repeats) can be used, then count their orders.

Call Central's players A, B, C and Northern's players X, Y, Z. There are 9 possible games $(AX, AY, \ldots, CZ)$, and each must be played exactly twice.

Step 1. Sort the six pairings into two families.

$$\begin{aligned}\text{Family 1:}\quad & \{AX, BY, CZ\},\ \{AY, BZ, CX\},\ \{AZ, BX, CY\}\\ \text{Family 2:}\quad & \{AX, BZ, CY\},\ \{AY, BX, CZ\},\ \{AZ, BY, CX\}\end{aligned}$$

Within each family the three pairings together contain all 9 games once each. A pairing from Family 1 and a pairing from Family 2 always have *exactly one* game in common (for example, $\{AX, BY, CZ\}$ and $\{AY, BX, CZ\}$ share only CZ; one can check all 9 combinations).

Step 2. Which collections of six rounds work? Pick any pairing p from Family 1 and any pairing q from Family 2, and let g be their one shared game. The game g appears in exactly one pairing of each family, namely p and q, so the number of times g is played is (uses of p) + (uses of q). This must be 2 for *every* such p and q. So all Family 1 pairings are used the same number of times, say a, and all Family 2 pairings are used $2 - a$ times. Conversely, any such choice plays every game exactly twice and uses $3a + 3(2 - a) = 6$ rounds.

Step 3. Count orders in each case.

- $a = 1$: all six pairings once each, in any order: $6! = 720$ ways.
- $a = 2$: the three Family 1 pairings, each twice: $\dfrac{6!}{2!\,2!\,2!} = \dfrac{720}{8} = 90$ ways.
- $a = 0$: the three Family 2 pairings, each twice: likewise 90 ways.

Total: $720 + 90 + 90 = 900$.

Watch out. It is easy to assume the six rounds must all be different. That counts only the $a = 1$ case and gives 720 — choice (C). Remembering just one of the "repeat" families gives $720 + 90 = 810$ — choice (D).

Another way. Follow players instead of rounds. Player A's list of opponents over the six rounds contains X, Y, Z twice each: $\frac{6!}{2!\,2!\,2!} = 90$ lists. Given A's list (say $XXYYZZ$), B's list also has each opponent twice and never repeats A's opponent in the same round; so rounds 1–2 avoid X, rounds 3–4 avoid Y,

rounds 5–6 avoid Z. If B opens with Y, Z in either order, then B's second game against Z must fall in rounds 3–4 and the second against Y in rounds 5–6, so rounds 3–4 are X, Z and rounds 5–6 are X, Y, each in either order: $2^3 = 8$ lists. Opening YY forces $YYZZXX$, and opening ZZ forces $ZZXXYY$. That is 10 lists. Then C must face whoever is left each round. So there are $90 \cdot 10 = 900$ schedules.

Answer: (E) 900

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24 Number Theory

Source: 2022 AMC 10B #25

Let $x_0, x_1, x_2, \ldots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}?$$

(A) 6 **(B)** 7 **(C)** 12 **(D)** 14 **(E)** 15

Key idea. S_n is just the number whose binary digits are $x_{n-1} \ldots x_1 x_0$, so the requested sum is a block of four binary digits: $\dfrac{S_{2023} - S_{2019}}{2^{2019}}$. The congruence plus the size bound $0 \leq S_n < 2^n$ pins S_n down exactly — a congruence modulo 2^n for a number smaller than 2^n leaves only a handful of options.

Step 1: the target in terms of S_n. Subtracting,

$$S_{2023} - S_{2019} = x_{2019} \cdot 2^{2019} + x_{2020} \cdot 2^{2020} + x_{2021} \cdot 2^{2021} + x_{2022} \cdot 2^{2022},$$

so the desired sum equals $\dfrac{S_{2023} - S_{2019}}{2^{2019}}$.

Step 2: an exact formula for S_n. Each x_k is 0 or 1, so $0 \leq S_n \leq 1 + 2 + \cdots + 2^{n-1} = 2^n - 1$. The congruence says $7S_n - 1$ is a multiple of 2^n; write $7S_n - 1 = m_n 2^n$. Since $7S_n \equiv 1$, $S_n \geq 1$, so $m_n \geq 0$; and $7S_n - 1 \leq 7(2^n - 1) - 1 < 7 \cdot 2^n$, so $m_n \leq 6$. Thus

$$7S_n = m_n 2^n + 1, \qquad m_n \in \{0, 1, \ldots, 6\}.$$

Step 3: find m_n modulo 7. The left side is a multiple of 7, so $m_n 2^n \equiv -1 \equiv 6 \pmod 7$. Powers of 2 modulo 7 cycle with period 3: $2^0 \equiv 1$, $2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv 1$.

- $2019 = 3 \cdot 673$, so $2^{2019} \equiv 1$ and $m_{2019} \equiv 6$, giving $m_{2019} = 6$.
- $2023 = 3 \cdot 674 + 1$, so $2^{2023} \equiv 2$ and $2m_{2023} \equiv 6$, giving $m_{2023} = 3$ (check: $2 \cdot 3 = 6$).

Hence $S_{2019} = \dfrac{6 \cdot 2^{2019} + 1}{7}$ and $S_{2023} = \dfrac{3 \cdot 2^{2023} + 1}{7}$.

Step 4: compute. The $+1$'s cancel:

$$\frac{S_{2023}-S_{2019}}{2^{2019}}=\frac{3\cdot 2^{2023}-6\cdot 2^{2019}}{7\cdot 2^{2019}}=\frac{3\cdot 2^4-6}{7}=\frac{48-6}{7}=6.$$

In binary, $6=0110_2$: $x_{2019}=0$, $x_{2020}=1$, $x_{2021}=1$, $x_{2022}=0$.

Another way. Compute a few terms: $S_1=1$, $S_3=7$, $S_4=7$, $S_5=23$, $S_6=55$, so the bits are $1,1,1,0,1,1,0,1,1,\ldots$ For $k\ge 1$, $x_k=0$ exactly when k is a multiple of 3. (This is no accident: by Step 2, $x_n=\frac{S_{n+1}-S_n}{2^n}=\frac{2m_{n+1}-m_n}{7}$, and by Step 3 the value of m_n depends only on n modulo 3, cycling $3,5,6$ for $n=1,2,3,\ldots$; this gives $x_n=1,1,0$ in turn.) Then $x_{2019}=0$, $x_{2020}=x_{2021}=1$, $x_{2022}=0$, and the sum is $0+2+4+0=6$.

Answer: (A) 6

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25 Counting & Probability

Source: 2012 AMC 10A #25

> Real numbers x, y, and z are chosen independently and at random from the interval $[0,n]$ for some positive integer n. The probability that no two of x, y, and z are within 1 unit of each other is greater than $\frac{1}{2}$. What is the smallest possible value of n?
>
> **(A)** 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Key idea. Three independent uniform numbers are a random point in the cube $[0,n]^3$, so the probability is a ratio of volumes. Sorting the three numbers first, and then sliding them together to "eat up" the required gaps, turns the good region into a smaller copy of the same shape.

Step 1: sort. Ties have probability 0, and each of the 6 orderings of x,y,z is equally likely by symmetry. So we may assume $z<y<x$ and compute the probability within that ordering; the answer is the same.

The region $0\le z\le y\le x\le n$ is one sixth of the cube, volume $\frac{n^3}{6}$.

Step 2: the good region. With $z\le y\le x$, "no two within 1 unit" means $y-z\ge 1$ and $x-y\ge 1$ (then automatically $x-z\ge 2$). Remove the gaps: let

$$u=z,\qquad v=y-1,\qquad w=x-2.$$

For $n\ge 2$ the conditions become $0\le u\le v\le w\le n-2$. (For $n=1$ no such triple fits in $[0,1]$, so the probability is 0.) This is a shift (it does not change volume), and it is the same kind of region with n replaced by $n-2$, so its volume is $\frac{(n-2)^3}{6}$.

Step 3: the probability.

$$P=\frac{(n-2)^3/6}{n^3/6}=\left(\frac{n-2}{n}\right)^3=\left(1-\frac{2}{n}\right)^3.$$

For $n\ge 2$ this increases as n increases, so we need the first n with $P>\frac{1}{2}$.

Step 4: test the boundary. For $n=9$: $\left(\frac{7}{9}\right)^3=\frac{343}{729}$, and $2\cdot 343=686<729$, so $P<\frac{1}{2}$. For $n=10$: $\left(\frac{8}{10}\right)^3=\frac{64}{125}$, and $2\cdot 64=128>125$, so $P>\frac{1}{2}$.

Since P grows with n, every $n \leq 9$ fails and $n = 10$ works. The smallest value is 10.

Watch out. The value $n = 9$ is very close: $\frac{343}{729} \approx 0.47$. Estimating $\frac{7}{9} \approx 0.8$ and cubing to get about 0.51 leads to choice (C). Compare $2(n-2)^3$ with n^3 exactly.

Answer: (D) 10