AMC 10

SET 21

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 13–23 of past AMC 10 contests (2000–2025) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**D**	2016 AMC 10B #13	Algebra
2	**C**	2011 AMC 10B #14	Geometry
3	**B**	2021 AMC 10B #15	Algebra
4	**D**	2022 AMC 10A #17	Number Theory
5	**C**	2007 AMC 10A #18	Geometry
6	**C**	2019 AMC 10B #19	Number Theory
7	**A**	2010 AMC 10A #21	Algebra
8	**C**	2008 AMC 10B #22	Counting & Probability
9	**E**	2022 AMC 10B #22	Geometry
10	**D**	2021 Fall AMC 10B #23	Counting & Probability

1 Algebra

Source: 2016 AMC 10B #13

> At Megapolis Hospital one year, multiple-birth statistics were as follows: Sets of twins, triplets, and quadruplets accounted for 1000 of the babies born. There were four times as many sets of triplets as sets of quadruplets, and there were three times as many sets of twins as sets of triplets. How many of these 1000 babies were in sets of quadruplets?
>
> **(A)** 25 **(B)** 40 **(C)** 64 **(D)** 100 **(E)** 160

Key idea. Everything is described relative to the number of sets of quadruplets, so call that q and express the rest in terms of it. Then count *babies*, not sets: each set of twins is 2 babies, and so on.

Step 1: count the sets. Let q be the number of sets of quadruplets. There are four times as many sets of triplets, so $4q$ sets of triplets. There are three times as many sets of twins as sets of triplets, so $3 \cdot 4q = 12q$ sets of twins.

Step 2: count the babies. Twins contribute $2 \cdot 12q = 24q$ babies, triplets contribute $3 \cdot 4q = 12q$ babies, and quadruplets contribute $4 \cdot q = 4q$ babies. Altogether

$$24q + 12q + 4q = 40q = 1000,$$

so $q = 25$.

Step 3: answer the actual question. There are 25 *sets* of quadruplets, and each set has 4 babies, so $4 \cdot 25 = 100$ babies were in sets of quadruplets.

Check: $12 \cdot 25 = 300$ sets of twins (600 babies), 100 sets of triplets (300 babies), 25 sets of quadruplets (100 babies); $600 + 300 + 100 = 1000$. ✓

Watch out. Stopping at $q = 25$ and choosing **(A)**. That is the number of *sets* of quadruplets; the question asks for babies.

Answer: (D) 100

2 Geometry

Source: 2011 AMC 10B #14

A rectangular parking lot has a diagonal of 25 meters and an area of 168 square meters. In meters, what is the perimeter of the parking lot?

(A) 52 **(B)** 58 **(C)** 62 **(D)** 68 **(E)** 70

Key idea. The perimeter needs only $l + w$, not l and w separately. The diagonal gives $l^2 + w^2$ and the area gives lw, and $(l + w)^2 = l^2 + 2lw + w^2$ combines exactly these two.

Step 1. Let the sides be l and w. By the Pythagorean Theorem, $l^2 + w^2 = 25^2 = 625$. The area gives $lw = 168$.

Step 2. Then

$$(l + w)^2 = l^2 + w^2 + 2lw = 625 + 2 \cdot 168 = 625 + 336 = 961 = 31^2.$$

Since $l + w > 0$, $l + w = 31$.

Step 3. The perimeter is $2(l + w) = 2 \cdot 31 = 62$.

Watch out. Stopping at $l + w = 31$. That is only half the perimeter (and is not a choice) — double it.

Another way. 25 is the hypotenuse of the 7–24–25 right triangle, and $7 \cdot 24 = 168$ matches the area. So the lot is 7 by 24, with perimeter $2(7 + 24) = 62$.

Answer: (C) 62

3 Algebra

Source: 2021 AMC 10B #15

The real number x satisfies the equation

$$x + \frac{1}{x} = \sqrt{5}.$$

What is the value of $x^{11} - 7x^7 + x^3$?

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** $\sqrt{5}$

Key idea. The exponents $11, 7, 3$ are spaced evenly around 7, so factor out x^7: the expression becomes $x^7\left(x^4 - 7 + \frac{1}{x^4}\right)$. Expressions like $x^4 + \frac{1}{x^4}$ come from squaring $x + \frac{1}{x}$ repeatedly.

Step 1: factor. Computing x^{11} directly from $x = \frac{\sqrt{5}\pm 1}{2}$ would be painful, so look for structure instead. The exponents 11 and 3 are each 4 away from 7, so pulling out x^7 leaves powers x^4 and x^{-4}:

$$x^{11} - 7x^7 + x^3 = x^7\left(x^4 - 7 + \frac{1}{x^4}\right).$$

(Check by multiplying back: $x^7 \cdot x^4 = x^{11}$ and $x^7 \cdot \frac{1}{x^4} = x^3$.) So we need $x^4 + \frac{1}{x^4}$, a symmetric expression of exactly the kind that squaring $x + \frac{1}{x}$ produces.

Step 2: square once. Squaring $x + \frac{1}{x} = \sqrt{5}$ gives

$$x^2 + 2 + \frac{1}{x^2} = 5, \qquad \text{so} \qquad x^2 + \frac{1}{x^2} = 3.$$

(The middle term 2 is $2 \cdot x \cdot \frac{1}{x}$.)

Step 3: square again.

$$x^4 + 2 + \frac{1}{x^4} = 9, \qquad \text{so} \qquad x^4 + \frac{1}{x^4} = 7.$$

(Again the middle term is $2 \cdot x^2 \cdot \frac{1}{x^2} = 2$.)

Step 4. Then $x^4 - 7 + \frac{1}{x^4} = 7 - 7 = 0$, and the expression equals $x^7 \cdot 0 = 0$. The 7 in the problem was chosen precisely to cancel $x^4 + \frac{1}{x^4}$; spotting that is the whole problem.

Such a real x does exist: $x^2 - \sqrt{5}\,x + 1 = 0$ has discriminant $5 - 4 = 1 > 0$, giving $x = \frac{\sqrt{5} \pm 1}{2}$. Both values are nonzero, so dividing by x was safe, and both give the same answer 0.

Watch out. Squaring without the cross term, as if $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2}$, gives 5 and then 25 instead of 3 and 7, and the expression no longer collapses. Always include the $+2$.

Answer: (B) 0

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4 Number Theory

Source: 2022 AMC 10A #17

How many three-digit positive integers $\underline{a}\ \underline{b}\ \underline{c}$ are there whose nonzero digits a, b, and c satisfy

$$0.\overline{\underline{a}\ \underline{b}\ \underline{c}} = \frac{1}{3}(0.\overline{a} + 0.\overline{b} + 0.\overline{c})?$$

(The bar indicates repetition, thus $0.\overline{\underline{a}\ \underline{b}\ \underline{c}}$ is the infinite repeating decimal $0.\underline{a}\ \underline{b}\ \underline{c}\ \underline{a}\ \underline{b}\ \underline{c}\cdots$)

(A) 9 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 14

Key idea. Repeating decimals are really fractions: a one-digit repeat is that digit over 9, a three-digit repeat is that number over 999. Converting turns the strange-looking condition into a plain linear equation in the digits, and since $999 = 27 \cdot 37$, the denominators clear nicely.

Step 1. We have $0.\overline{\underline{a}\ \underline{b}\ \underline{c}} = \dfrac{100a + 10b + c}{999}$ and $0.\overline{a} = \dfrac{a}{9}$, $0.\overline{b} = \dfrac{b}{9}$, $0.\overline{c} = \dfrac{c}{9}$. The condition becomes

$$\frac{100a + 10b + c}{999} = \frac{a + b + c}{27}.$$

Step 2. Multiply by $999 = 27 \cdot 37$: $100a + 10b + c = 37(a + b + c) = 37a + 37b + 37c$. Collecting terms, $63a = 27b + 36c$, and dividing by 9,

$$7a = 3b + 4c, \qquad \text{or} \qquad 3(b - a) = 4(a - c).$$

Step 3. If $a = b = c$, both sides are 0, so all nine numbers $111, 222, \ldots, 999$ work. Otherwise $b - a$ must be a multiple of 4 and $a - c$ a multiple of 3: write $b - a = 4t$ and $a - c = 3t$ with $t \neq 0$. Digits differ by at most 8, so $|t| \leq 2$.

- $t = 1$: $b = a + 4$, $c = a - 3$, needing $4 \leq a \leq 5$: 481 and 592.
- $t = -1$: $b = a - 4$, $c = a + 3$, needing $5 \leq a \leq 6$: 518 and 629.
- $t = \pm 2$: $|b - a| = 8$ forces $\{a, b\} = \{1, 9\}$, and then $c = a \mp 6$ is -5 or 15, not a digit.

For example, $7 \cdot 4 = 28 = 3 \cdot 8 + 4 \cdot 1$ confirms 481. In total there are $9 + 4 = 13$ numbers.

Watch out. Spotting only the repdigits $111, \ldots, 999$ (where the equation is obviously true) gives 9 — choice (A).

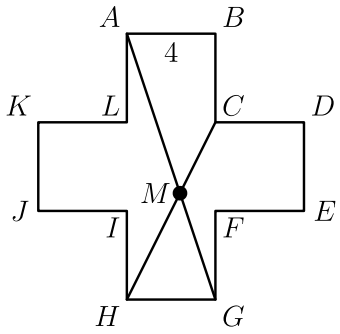
Answer: (D) 13

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5 Geometry

Source: 2007 AMC 10A #18

Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 4, and each two consecutive sides form a right angle. Suppose that $\overline{AG}$ and $\overline{CH}$ meet at M. What is the area of quadrilateral $ABCM$?



(A) $\frac{44}{3}$ **(B)** 16 **(C)** $\frac{88}{5}$ **(D)** 20 **(E)** $\frac{62}{3}$

Key idea. B, C, G all lie on one vertical line, so $ABCM$ is the big triangle ABG with triangle CMG cut away. Everything then hinges on how far M is from that line, and the two parallel vertical segments $\overline{AH}$ and $\overline{CG}$ point to similar triangles.

Step 1: the frame. The cross is three 4×4 squares tall, so A, L, I, H lie on one vertical line, and B, C, F, G on another, 4 units to the right. Thus $ABGH$ is a 4×12 rectangle, with $AH = 12$, $CG = 8$ (C is 4 below B and G is 12 below B), and $AB = 4$.

Step 2: locate M. Since $\overline{AH} \parallel \overline{CG}$, triangles MAH and MGC are similar (vertical angles at M, alternate angles along the parallels). Their ratio is $AH : GC = 12 : 8 = 3 : 2$, so the distances from M to the two vertical lines are in ratio $3 : 2$. They add to 4, so M is $\frac{2}{5} \cdot 4 = \frac{8}{5}$ from line BG.

Step 3: areas. Triangle ABG has legs $AB = 4$ and $BG = 12$: area $\frac{1}{2} \cdot 4 \cdot 12 = 24$. Triangle CMG has base $CG = 8$ on line BG and height $\frac{8}{5}$: area $\frac{1}{2} \cdot 8 \cdot \frac{8}{5} = \frac{32}{5}$. Hence

$$[ABCM] = 24 - \frac{32}{5} = \frac{88}{5}.$$

Another way. Coordinates with the center at the origin: $A = (-2, 6)$, $B = (2, 6)$, $C = (2, 2)$, $G = (2, -6)$, $H = (-2, -6)$. Line AG is $y = -3x$ and line CH is $y = 2x - 2$, meeting at $M = \left(\frac{2}{5}, -\frac{6}{5}\right)$. The shoelace formula on A, B, C, M gives $\frac{88}{5}$.

Watch out. The picture tempts you to put M at the center of the cross. The center does lie on $\overline{AG}$, but not on $\overline{CH}$; using it anyway gives $24 - \frac{1}{2} \cdot 8 \cdot 2 = 16$, choice (B).

Answer: (C) $\frac{88}{5}$

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6 Number Theory

Source: 2019 AMC 10B #19

Let S be the set of all positive integer divisors of 100,000. How many numbers are the product of two distinct elements of S?

(A) 98 **(B)** 100 **(C)** 117 **(D)** 119 **(E)** 121

Key idea. Divisors of $100{,}000 = 2^5 \cdot 5^5$ are described by a pair of exponents, so products become *sums* of exponents. Count all possible exponent sums, then remove the few that can only be made by multiplying a divisor by itself.

Step 1. Since $100{,}000 = 10^5 = 2^5 \cdot 5^5$, every element of S is $2^a 5^b$ with $0 \le a \le 5$ and $0 \le b \le 5$ (so $|S| = 36$). A product of two elements is

$$2^a 5^b \cdot 2^c 5^d = 2^{a+c}\, 5^{b+d}, \qquad 0 \le a + c \le 10, \quad 0 \le b + d \le 10.$$

So every product is one of the $11 \cdot 11 = 121$ numbers $2^s 5^t$ with $0 \le s \le 10$ and $0 \le t \le 10$ (different exponent pairs give different numbers).

Step 2. Which of these 121 numbers $2^s 5^t$ can be made from two *distinct* divisors? If $1 \le s \le 9$, we can split $s = a + c$ with $a \ne c$ and $0 \le a, c \le 5$ (for instance $s = 1 = 0 + 1$, $s = 9 = 4 + 5$; in general take $a = \lfloor s/2 \rfloor$, $c = \lceil s/2 \rceil$ when s is odd and $a = s/2 - 1$, $c = s/2 + 1$ when s is even). Choose any $b + d = t$ with $0 \le b, d \le 5$; since $a \ne c$, the two divisors are different. The same works when $1 \le t \le 9$, swapping the roles of the two primes.

Step 3. If both $s \in \{0, 10\}$ and $t \in \{0, 10\}$, the splits are forced: $s = 0$ needs $a = c = 0$, $s = 10$ needs $a = c = 5$, and likewise for t. Then the two divisors are equal, so these four numbers,

$$1, \quad 2^{10} = 1024, \quad 5^{10}, \quad 10^{10},$$

cannot be made from distinct elements.

Step 4. The count is $121 - 4 = 117$.

Watch out. Ignoring the word *distinct* gives 121, choice (E). Removing only the obvious squares $1 = 1 \cdot 1$ and $10^{10} = 100{,}000 \cdot 100{,}000$ gives 119, choice (D); $2^{10} = 32 \cdot 32$ and $5^{10} = 3125 \cdot 3125$ are also forced squares.

Answer: (C) 117

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7 Algebra

Source: 2010 AMC 10A #21

The polynomial

$$x^3 - ax^2 + bx - 2010$$

has three positive integer roots. What is the smallest possible value of a?

(A) 78 **(B)** 88 **(C)** 98 **(D)** 108 **(E)** 118

Key idea. A polynomial with given roots is pinned down by Vieta's formulas: the constant term gives the *product* of the roots, and a is their *sum*. So the question becomes: split 2010 into three positive factors with the smallest possible sum.

Step 1. Let the roots be the positive integers r, s, t. Then

$$x^3 - ax^2 + bx - 2010 = (x - r)(x - s)(x - t),$$

and comparing coefficients gives $r + s + t = a$ and $rst = 2010$.

Step 2. Factor: $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. The prime 67 must divide one of the roots, say t. Then $t = 67$ or $t \geq 2 \cdot 67 = 134$.

Step 3. If $t \geq 134$, then already $a > 134$, far more than the value 78 we reach below.

If $t = 67$, then $rs = 30$, and we minimise $r + s$ over the factor pairs of 30:

$$1 + 30 = 31, \quad 2 + 15 = 17, \quad 3 + 10 = 13, \quad 5 + 6 = 11.$$

The smallest is $5 + 6 = 11$, giving $a = 5 + 6 + 67 = 78$.

Step 4. This value is actually achieved: with roots $5, 6, 67$,

$$(x - 5)(x - 6)(x - 67) = x^3 - 78x^2 + 767x - 2010,$$

which has the required form with $a = 78$, $b = 767$. Every other factorisation gives a larger sum, so the minimum is 78.

Watch out. Writing $2010 = 30 \cdot 67$ and using 1 as the third root gives roots $1, 30, 67$ and $a = 98$, choice (C). Once $t = 67$ is fixed, the other two roots should be as close together as possible: for two factors with a fixed product, the closer they are, the smaller their sum.

Answer: (A) 78

8 Counting & Probability

Source: 2008 AMC 10B #22

Three red beads, two white beads, and one blue bead are placed in a line in random order. What is the probability that no two neighboring beads are the same color?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

Key idea. Count color patterns. The three reds are the hardest to keep apart, so place them first in non-adjacent spots; then see how the two whites and the blue can fill the gaps without two whites touching.

Step 1. Total patterns. Number the positions 1 to 6. A color pattern is fixed by choosing where the 3 reds go and then where the blue goes among the other 3 spots:

$$\binom{6}{3} \cdot 3 = 20 \cdot 3 = 60 = \frac{6!}{3!\,2!\,1!}.$$

All 60 patterns are equally likely.

Step 2. Place the reds. No two reds may touch. The sets of three pairwise non-adjacent positions in 1–6 are

$$\{1,3,5\}, \quad \{2,4,6\}, \quad \{1,3,6\}, \quad \{1,4,6\}.$$

(These are all: if the first red is at 1, the other two come from $3,4,5,6$ with a gap — $\{3,5\}, \{3,6\}, \{4,6\}$; if the first red is at 2, the only option is $\{4,6\}$; a first red at 3 or later leaves no room.)

Step 3. Place white, white, blue.

- Reds at $\{1,3,5\}$: the free spots $2,4,6$ are pairwise apart, so blue can go in any of them — 3 ways.
- Reds at $\{2,4,6\}$: free spots $1,3,5$, again 3 ways.
- Reds at $\{1,3,6\}$: free spots $2,4,5$. Spots 4 and 5 touch, so they cannot both be white; blue must be at 4 or 5 — 2 ways.
- Reds at $\{1,4,6\}$: free spots $2,3,5$. Spots 2 and 3 touch, so blue must be at 2 or 3 — 2 ways.

That is $3+3+2+2=10$ good patterns.

Step 4. The probability is $\frac{10}{60} = \frac{1}{6}$.

Watch out. It is tempting to count only the perfectly alternating red layouts $\{1,3,5\}$ and $\{2,4,6\}$. That gives $\frac{6}{60} = \frac{1}{10}$ — choice (B). The layouts $\{1,3,6\}$ and $\{1,4,6\}$, with reds at both ends, add 4 more.

Answer: (C) $\frac{1}{6}$

9 Geometry

Source: 2022 AMC 10B #22

Let S be the set of circles in the coordinate plane that are tangent to each of the three circles with equations

$$x^2+y^2=4, \qquad x^2+y^2=64, \qquad \text{and} \qquad (x-5)^2+y^2=3.$$

What is the sum of the areas of all circles in S?

(A) 48π **(B)** 68π **(C)** 96π **(D)** 102π **(E)** 136π

Key idea. Two of the given circles share the center $O=(0,0)$. A circle tangent to two concentric circles has its radius and its distance from O completely pinned down, so handle those two first; the third circle then decides how many positions each size has.

Step 1: tangent to the outer circle. Let a circle in S have radius r and center P with $OP=d$. The given circles have radii 2 and 8 about O; the small one lies strictly inside the big one. A circle outside the big circle, or one enclosing it, could never touch the small circle. So our circle lies inside the big circle and touches it from within:

$$d+r=8.$$

Step 2: tangent to the inner circle. Our circle reaches out to distance 8 from O, so it is not inside the radius-2 circle. Either it touches it from outside, $d-r=2$, or it encloses it, $r-d=2$.

- $d+r=8$, $d-r=2$: $r=3$, $d=5$.
- $d+r=8$, $r-d=2$: $r=5$, $d=3$.

So every circle in S has radius 3 with center 5 from O, or radius 5 with center 3 from O.

Step 3: tangent to the third circle. The third circle has center $Q=(5,0)$ and radius $\sqrt{3}$ (careful: the radius is $\sqrt{3}$, not 3). Since $r>\sqrt{3}$, tangency means $PQ=q$ with $q=r+\sqrt{3}$ (touching from outside) or $q=r-\sqrt{3}$ (enclosing it). So P lies on the circle of radius d about O and on the circle of radius q about Q, where $OQ=5$. Two circles meet in two points exactly when $|d-q|<5<d+q$.

For $(r,d)=(3,5)$ and $q=3\pm\sqrt{3}$: $|d-q|=2\mp\sqrt{3}$ and $d+q=8\pm\sqrt{3}$. For $(r,d)=(5,3)$ and $q=5\pm\sqrt{3}$: $|d-q|=2\pm\sqrt{3}$ and $d+q=8\pm\sqrt{3}$. In all four cases $|d-q|\le 2+\sqrt{3}<4<5$ and $d+q\ge 8-\sqrt{3}>6>5$, so each case gives two centers, mirror images in the x-axis.

Step 4. That is 4 circles of radius 3 and 4 circles of radius 5, with total area

$$4\cdot 9\pi+4\cdot 25\pi=36\pi+100\pi=136\pi.$$

Watch out. Finding one center per case and forgetting its reflection in the x-axis gives $2\cdot 9\pi+2\cdot 25\pi=68\pi$, choice (B).

Answer: (E) 136π

10 Counting & Probability

Source: 2021 Fall AMC 10B #23

Each of the 5 sides and the 5 diagonals of a regular pentagon are randomly and independently colored red or blue with equal probability. What is the probability that there will be a triangle whose vertices are among the vertices of the pentagon such that all of its sides have the same color?

(A) $\frac{2}{3}$ **(B)** $\frac{105}{128}$ **(C)** $\frac{125}{128}$ **(D)** $\frac{253}{256}$ **(E)** 1

Key idea. "There will be a one-colored triangle" is almost certain, so count the opposite: colorings of the 10 segments with *no* one-colored triangle. Those turn out to be very rigid, which is what makes the count small.

Step 1: setup. The 5 sides and 5 diagonals are all $\binom{5}{2} = 10$ segments joining the 5 vertices, and every triple of vertices gives a triangle. Each segment has 2 colors, so there are $2^{10} = 1024$ equally likely colorings.

Step 2: each vertex has two of each color. Suppose a coloring has no one-colored triangle, and some vertex V has 3 segments of the same color, say red, to X, Y, Z. If any of XY, YZ, ZX were red, it would form a red triangle with V. So all three are blue — but then XYZ is a blue triangle. This is impossible, so each vertex has at most 2 red and at most 2 blue segments. Each vertex has 4 segments, so exactly 2 of each.

Step 3: the red segments form a 5-cycle. Every vertex touches exactly 2 red segments, so following red segments from vertex to vertex we travel around closed loops, each using at least 3 vertices. The 5 vertices cannot split into two such loops ($3 + 2$ fails), so the red segments form one loop through all 5 vertices. The blue segments are the other 5, which also form a loop through all 5 vertices (like a pentagon and its star).

Conversely, such a coloring works: a 5-vertex loop contains no triangle, so neither color has one.

Step 4: count. A loop through 5 labeled vertices can be listed starting at any of 5 vertices in either of 2 directions, so the number of loops is $\frac{5!}{5 \cdot 2} = 12$. Hence 12 colorings avoid a one-colored triangle, and

$$P = 1 - \frac{12}{1024} = 1 - \frac{3}{256} = \frac{253}{256}.$$

Answer: (D) $\frac{253}{256}$