AMC 10

SET 23

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 13–23 of past AMC 10 contests (2000–2025) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**B**	2021 AMC 10B #13	Number Theory
2	**D**	2022 AMC 10A #15	Geometry
3	**C**	2010 AMC 10B #15	Algebra
4	**C**	2020 AMC 10B #17	Counting & Probability
5	**B**	2018 AMC 10B #17	Geometry
6	**E**	2024 AMC 10A #19	Algebra
7	**E**	2021 Fall AMC 10B #21	Counting & Probability
8	**E**	2014 AMC 10A #21	Algebra
9	**D**	2009 AMC 10A #22	Counting & Probability
10	**E**	2014 AMC 10B #23	Geometry

1 Number Theory

Source: 2021 AMC 10B #13

> Let n be a positive integer and d be a digit such that the value of the numeral $\underline{32d}$ in base n equals 263, and the value of the numeral $\underline{324}$ in base n equals the value of the numeral $\underline{11d1}$ in base six. What is $n+d$?
>
> **(A)** 10 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 16

Key idea. Write each numeral as a sum of powers of its base. The two base-n numerals $\underline{32d}$ and $\underline{324}$ differ only in the last digit, so subtracting the two equations wipes out every n at once.

Step 1. In base n, $\underline{32d} = 3n^2 + 2n + d$, so

$$3n^2 + 2n + d = 263.$$

Step 2. In base n, $\underline{324} = 3n^2 + 2n + 4$. In base six,

$$\underline{11d1} = 1 \cdot 6^3 + 1 \cdot 6^2 + d \cdot 6 + 1 = 216 + 36 + 6d + 1 = 253 + 6d.$$

So $3n^2 + 2n + 4 = 253 + 6d$.

Step 3. Subtract the second equation from the first: $d - 4 = 263 - (253 + 6d) = 10 - 6d$. Hence $7d = 14$ and $d = 2$ (a legal digit in base six).

Step 4. Now $3n^2 + 2n + 2 = 263$, so $3n^2 + 2n = 261$, i.e. $n(3n + 2) = 261 = 9 \cdot 29$. Trying $n = 9$: $9 \cdot 29 = 261$. Since $n(3n + 2)$ grows with n, this is the only positive solution. (Digit 4 requires $n > 4$, and $n = 9$ works.)

Therefore $n + d = 9 + 2 = 11$.

Answer: (B) 11

2 Geometry

Source: 2022 AMC 10A #15

> Quadrilateral $ABCD$ with side lengths $AB = 7, BC = 24, CD = 20, DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where a, b, and c are positive integers such that a and c have no common prime factor. What is $a + b + c$?
>
> **(A)** 260 **(B)** 855 **(C)** 1235 **(D)** 1565 **(E)** 1997

Key idea. We need the circle's radius, and the side lengths are suspiciously Pythagorean: $7^2 + 24^2 = 625 = 25^2$ and $15^2 + 20^2 = 625 = 25^2$. That hints that diagonal $\overline{AC}$ has length 25 and is a diameter. Prove it with the Law of Cosines and the cyclic-quadrilateral fact $\angle B + \angle D = 180^\circ$.

Step 1. Diagonal $\overline{AC}$ splits $ABCD$ into $\triangle ABC$ and $\triangle ADC$. By the Law of Cosines in each,

$$AC^2 = 7^2 + 24^2 - 2 \cdot 7 \cdot 24 \cos B = 625 - 336 \cos B,$$

$$AC^2 = 15^2 + 20^2 - 2 \cdot 15 \cdot 20 \cos D = 625 - 600 \cos D.$$

Because $ABCD$ is cyclic, $\angle B + \angle D = 180^\circ$, so $\cos D = -\cos B$. Setting the two expressions equal gives $-336 \cos B = 600 \cos B$, so $\cos B = 0$.

Step 2. Thus $\angle B = \angle D = 90^\circ$ and $AC = 25$. An inscribed right angle stands on a diameter, so $\overline{AC}$ is a diameter, the radius is $\frac{25}{2}$, and the circle's area is $\pi \left(\frac{25}{2}\right)^2 = \frac{625\pi}{4}$.

Step 3. The quadrilateral is two right triangles with legs $7, 24$ and $15, 20$:

$$[ABCD] = \tfrac{1}{2} \cdot 7 \cdot 24 + \tfrac{1}{2} \cdot 15 \cdot 20 = 84 + 150 = 234.$$

Step 4. The region inside the circle but outside $ABCD$ has area

$$\frac{625\pi}{4} - 234 = \frac{625\pi - 936}{4}.$$

Here $625 = 5^4$ and $4 = 2^2$ share no prime factor, so $a = 625$, $b = 936$, $c = 4$, and $a + b + c = 1565$.

Watch out. Taking $AC = 25$ as the *radius* instead of the diameter gives $625\pi - 234$, so $a+b+c = 860$, which is not a choice. The diameter is the hypotenuse of the inscribed right triangle.

Answer: (D) 1565

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3 Algebra

Source: 2010 AMC 10B #15

On a 50-question multiple choice math contest, students receive 4 points for a correct answer, 0 points for an answer left blank, and -1 point for an incorrect answer. Jesse's total score on the contest was 99. What is the maximum number of questions that Jesse could have answered correctly?

(A) 25 **(B)** 27 **(C)** 29 **(D)** 31 **(E)** 33

Key idea. Every extra correct answer beyond 25 must be "paid for" with wrong answers to keep the score at 99, and wrong answers use up questions. The 50-question limit is what stops us, so write it down as an inequality.

Step 1. Let x be the number correct and y the number incorrect. The score gives $4x - y = 99$, so $y = 4x - 99$. The number answered cannot exceed 50: $x + y \le 50$.

Step 2. Substitute: $x + (4x - 99) \le 50$, so $5x \le 149$ and $x \le 29.8$. Since x is a whole number, $x \le 29$.

Step 3. Check that 29 is actually possible: $y = 4 \cdot 29 - 99 = 116 - 99 = 17$ wrong, and $29 + 17 = 46 \le 50$, leaving 4 blank. The score is $116 - 17 = 99$. One more correct, $x = 30$, would need $y = 21$ and 51 questions — too many.

Watch out. The first combination most people find is 25 correct and 1 wrong ($100 - 1 = 99$), choice (A). It is valid, but not the maximum: trading in more questions as wrong answers lets x climb to 29.

Answer: (C) 29

4 Counting & Probability

Source: 2020 AMC 10B #17

> There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9 people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?
>
> **(A)** 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Key idea. Every pair is either two neighbors or two people directly across. Sort the pairings by *how many across-pairs* they use: once those are fixed, the rest of the people must pair with neighbors, and that is easy to count.

Number the people $0, 1, \ldots, 9$ around the circle. The five across-pairs are

$$\{0,5\}, \quad \{1,6\}, \quad \{2,7\}, \quad \{3,8\}, \quad \{4,9\};$$

call the across-pair $\{i, i+5\}$ "diameter i."

Key fact. After some diameters are used, the people left over fall into stretches of consecutive people along the circle. Within a stretch, neighbors must pair up, which works only if the stretch has an even number of people, and then in exactly one way (pair the first two, the next two, and so on).

No diameters. All 10 people pair with neighbors around the full circle: either $\{0,1\},\{2,3\},\ldots$ or $\{1,2\},\{3,4\},\ldots,\{9,0\}$. That is 2 ways.

One diameter. Choose it in 5 ways. Removing, say, 0 and 5 leaves the stretches $1,2,3,4$ and $6,7,8,9$, each with 4 people, so each pairs up in one way. That is 5 ways.

Two or four diameters. Suppose the used diameters are $i_1 < \cdots < i_k$ (numbers from 0 to 4). Walking around the circle, the removed people are $i_1, \ldots, i_k, i_1+5, \ldots, i_k+5$. The stretch between i_j and i_{j+1} has $i_{j+1} - i_j - 1$ people, and the stretch between i_k and i_1+5 has $i_1 + 5 - i_k - 1$ people. For all of these to be even, each of the k jumps

$$i_2 - i_1, \ \ldots, \ i_k - i_{k-1}, \ i_1 + 5 - i_k$$

must be odd. These jumps add up to exactly 5, an odd number, and k odd numbers add up to an odd total only when k is odd. So $k = 2$ and $k = 4$ are impossible.

Three diameters. Now the three jumps are odd and add up to 5, so they are $1,1,3$ in some order: the three chosen diameters are consecutive when 0–4 are read around in a cycle (so 4 is followed by 0), like $\{0,1,2\}$ or $\{4,0,1\}$. There are 5 such choices. For $\{0,1,2\}$, the leftover stretches are $3,4$ and $8,9$, which pair in one way. That is 5 ways.

Five diameters. Everyone pairs across: 1 way.

Total. $2 + 5 + 5 + 1 = 13$.

Watch out. The full circle of 10 people has *two* all-neighbor pairings, not one — you can start the pairing at person 0 or at person 1. Counting only one gives $1 + 5 + 5 + 1 = 12$, choice (B).

Answer: (C) 13

5 Geometry

Source: 2018 AMC 10B #17

In rectangle $PQRS$, $PQ = 8$ and $QR = 6$. Points A and B lie on PQ, points C and D lie on QR, points E and F lie on RS, and points G and H lie on SP so that $AP = BQ < 4$ and the convex octagon $ABCDEFGH$ is equilateral. The length of a side of this octagon can be expressed in the form $k + m\sqrt{n}$, where k, m, and n are integers and n is not divisible by the square of any prime. What is $k + m + n$?

(A) 1 **(B)** 7 **(C)** 21 **(D)** 92 **(E)** 106

Key idea. The octagon is the rectangle with its four corners cut off as right triangles. Name the legs of those triangles, then write "every side has the same length s" as equations — the slanted sides via the Pythagorean theorem.

Step 1: The four cut corners are congruent. Let s be the side length and $x = AP = BQ$. Since $PQ = AP + AB + BQ = 2x + s = 8$, we get $x = \frac{8-s}{2}$.

Right triangles APH and BQC both have hypotenuse s and a leg x, so their other legs are equal: $PH = QC = y$, say. On side QR, $DR = 6 - y - s$; on side SP, $GS = 6 - y - s$ as well. So the right triangles at R and S have hypotenuse s and equal legs $DR = GS$, hence $RE = SF$. As $RE + EF + FS = 8$ with $EF = s$, each equals $\frac{8-s}{2} = x$, and then $DR = GS = \sqrt{s^2 - x^2} = y$. So all four corner triangles have legs x (along the long sides) and y (along the short sides).

Step 2: Equal sides. The horizontal sides, vertical sides, and slanted sides have lengths $8 - 2x$, $6 - 2y$, and $\sqrt{x^2 + y^2}$, so

$$8 - 2x = 6 - 2y = \sqrt{x^2 + y^2}.$$

The first equality gives $y = x - 1$. Squaring the outer equality:

$$(8 - 2x)^2 = x^2 + (x - 1)^2 \implies 64 - 32x + 4x^2 = 2x^2 - 2x + 1,$$

so $2x^2 - 30x + 63 = 0$.

Step 3: Solve. The quadratic formula gives $x = \dfrac{30 \pm \sqrt{900 - 504}}{4} = \dfrac{30 \pm 6\sqrt{11}}{4} = \dfrac{15 \pm 3\sqrt{11}}{2}$. Since $x < 4$, take the minus sign: $x = \frac{15-3\sqrt{11}}{2} \approx 2.53$, and then $y = x - 1 \approx 1.53 > 0$, so the octagon really exists.

Step 4. The side is $s = 8 - 2x = 8 - (15 - 3\sqrt{11}) = -7 + 3\sqrt{11} \approx 2.95$. So $k = -7$, $m = 3$, $n = 11$, and

$$k + m + n = -7 + 3 + 11 = 7.$$

Watch out. The natural way to write the side is $3\sqrt{11} - 7$, and it is tempting to read off 7, 3, 11 and add them to get 21, choice (C). The form is $k + m\sqrt{n}$, so the constant term is $k = -7$, not 7.

Answer: (B) 7

6 Algebra

Source: 2024 AMC 10A #19

> The first three terms of a geometric sequence are the integers a, 720, and b, where $a < 720 < b$. What is the sum of the digits of the least possible value of b?
>
> **(A)** 9 **(B)** 12 **(C)** 16 **(D)** 18 **(E)** 21

Key idea. In a geometric sequence the middle term squared equals the product of its neighbours, so $ab = 720^2$. That turns the question into number theory: b must be a *divisor* of 720^2, and we want the smallest such divisor above 720.

Step 1. The key equation. If the common ratio is r, then $720 = ar$ and $b = 720r$, so

$$ab = \frac{720}{r} \cdot 720r = 720^2.$$

Also $r > 0$: a negative ratio would make b negative, but $b > 720$. So a and b are positive.

Step 2. Translate. Since $a = \dfrac{720^2}{b}$ must be an integer, b must divide 720^2. Conversely, if b is any divisor of 720^2 with $b > 720$, then $a = \dfrac{720^2}{b}$ is a positive integer less than 720, and $a, 720, b$ is geometric. So we need the *smallest divisor of* 720^2 *that exceeds* 720.

Step 3. Search just above 720. Factor: $720 = 2^4 \cdot 3^2 \cdot 5$, so

$$720^2 = 2^8 \cdot 3^4 \cdot 5^2.$$

A divisor can use only the primes $2, 3, 5$, with at most eight 2s, four 3s and two 5s. The numbers from 721 to 768 built only from 2, 3, 5 are

$$729 = 3^6, \qquad 750 = 2 \cdot 3 \cdot 5^3, \qquad 768 = 2^8 \cdot 3.$$

729 has too many 3s and 750 has too many 5s, but $768 = 2^8 \cdot 3$ fits.

Step 4. Finish. So the least b is 768, with ratio $\frac{768}{720} = \frac{16}{15}$ and $a = \frac{720^2}{768} = 675$. Check: $675 \cdot \frac{16}{15} = 720$ and $720 \cdot \frac{16}{15} = 768$. ✓

The digit sum is $7 + 6 + 8 = 21$.

Watch out. The ratio does not have to be an integer. Assuming it does, the smallest choice is $r = 2$, giving $b = 1440$ and digit sum $1 + 4 + 4 + 0 = 9$ — choice (A). Fractional ratios such as $\frac{16}{15}$ give much smaller b.

Answer: (E) 21

7 Counting & Probability

Source: 2021 Fall AMC 10B #21

Regular polygons with 5, 6, 7, and 8 sides are inscribed in the same circle. No two of the polygons share a vertex, and no three of their sides intersect at a common point. At how many points inside the circle do two of their sides intersect?

(A) 52 **(B)** 56 **(C)** 60 **(D)** 64 **(E)** 68

Key idea. Two sides of the *same* convex polygon never cross, so every intersection point comes from a pair of different polygons. There are only $\binom{4}{2} = 6$ such pairs, so count the crossings for one pair, a k-gon and an n-gon with $k < n$, and add. The picture to hold in mind: each side of the smaller polygon cuts a thin cap off the circle, and the larger polygon must poke into that cap.

Step 1: when do two chords cross? All sides are chords of the circle. Two chords with four different endpoints cross (inside the circle) exactly when their endpoints alternate around the circle: one endpoint of the second chord lies on each of the two arcs cut off by the first chord. No polygons share a vertex, so all endpoints are different.

Step 2: one side of the smaller polygon. Take a side s of the k-gon. It cuts off an arc of $\frac{360^\circ}{k}$, while consecutive vertices of the n-gon are $\frac{360^\circ}{n}$ apart. Since $k < n$, the arc is longer than that gap, so it contains at least one vertex of the n-gon; it cannot contain all of them (the rest of the circle is longer still). The n-gon's vertices on this arc are consecutive, forming a run that the boundary of the n-gon enters by one side and leaves by another.

By Step 1, a side of the n-gon crosses s exactly when one of its endpoints is on the arc and the other is off it. Going around the n-gon, this happens exactly twice: once entering the run and once leaving it. So each side of the k-gon meets the n-gon in exactly 2 points.

Step 3: one pair. The k-gon has k sides, so the pair contributes $2k$ crossings, where k is the number of sides of the *smaller* polygon.

Step 4: add over the six pairs. No three sides meet at a point, so all these crossings are different points.

- Pentagon with each of the 6-, 7-, 8-gons: $3 \cdot (2 \cdot 5) = 30$.
- Hexagon with each of the 7-, 8-gons: $2 \cdot (2 \cdot 6) = 24$.
- Heptagon with the octagon: $2 \cdot 7 = 14$.

The total is $30 + 24 + 14 = 68$.

Answer: (E) 68

8 Algebra

Source: 2014 AMC 10A #21

Positive integers a and b are such that the graphs of $y = ax + 5$ and $y = 3x + b$ intersect the x-axis at the same point. What is the sum of all possible x-coordinates of these points of intersection?

(A) -20 **(B)** -18 **(C)** -15 **(D)** -12 **(E)** -8

Key idea. A line meets the x-axis where $y = 0$. Setting the two x-intercepts equal turns the geometry into a single equation in the positive integers a and b, and a product equal to a fixed number has only finitely many solutions.

Step 1. Put $y = 0$ in each equation:

$$0 = ax + 5 \implies x = -\frac{5}{a}, \qquad 0 = 3x + b \implies x = -\frac{b}{3}.$$

Step 2. The intercepts are the same point, so

$$\frac{5}{a} = \frac{b}{3} \quad \Longrightarrow \quad ab = 15.$$

Step 3. With a and b positive integers, a must be a positive divisor of 15, and then $b = 15/a$. The pairs are

$$(a, b) = (1, 15),\ (3, 5),\ (5, 3),\ (15, 1).$$

Step 4. The common intercept $x = -5/a$ is then

$$-5, \quad -\frac{5}{3}, \quad -1, \quad -\frac{1}{3}.$$

These are four different points, and their sum is

$$-5 - 1 - \frac{5}{3} - \frac{1}{3} = -6 - \frac{6}{3} = -6 - 2 = -8.$$

Watch out. Nothing requires the x-coordinate to be an integer. Dropping $-\frac{5}{3}$ and $-\frac{1}{3}$ leaves a sum of -6, which is not a choice — a sign that something was lost.

Answer: (E) -8

9 Counting & Probability

Source: 2009 AMC 10A #22

> Two cubical dice each have removable numbers 1 through 6. The twelve numbers on the two dice are removed, put into a bag, then drawn one at a time and randomly reattached to the faces of the cubes, one number to each face. The dice are then rolled and the numbers on the two top faces are added. What is the probability that the sum is 7?
>
> **(A)** $\frac{1}{9}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{2}{11}$ **(E)** $\frac{1}{5}$

Key idea. The relabelling looks complicated, but the only thing that matters is which two of the twelve numbered tiles end up on top. Because the tiles were attached at random and the dice are rolled at random, those two tiles are simply two different tiles chosen at random from the bag — so think about tiles, not dice.

Step 1 (what is really random). Think of the twelve tiles as physical objects: two 1s, two 2s, ..., two 6s. After random attaching and rolling, the top face of the first die carries a tile, and every one of the twelve tiles is equally likely to be there. The top face of the second die carries a *different* tile, and given the first, each of the other eleven tiles is equally likely.

Step 2 (condition on the first die). Suppose the first die shows N. For the sum to be 7, the second die must show $7 - N$. Note that $7 - N \neq N$, because 7 is odd, so both tiles marked $7 - N$ are still among the remaining eleven.

Step 3. So, whatever N is, the chance the second tile is marked $7 - N$ is $\frac{2}{11}$. Hence the probability that the sum is 7 is $\frac{2}{11}$.

Check by counting. There are $12 \cdot 11 = 132$ equally likely ordered pairs of different tiles (first die's top, second die's top). For each of the 12 choices of first tile there are exactly 2 good partners, giving 24 good pairs, and $\frac{24}{132} = \frac{2}{11}$.

Compare with ordinary dice: there the second die is independent of the first and shows $7 - N$ with probability $\frac{1}{6} = \frac{2}{12}$, so the sum is 7 in 6 of the 36 outcomes. Here the second number is drawn from a bag that is missing one copy of N — the one number that can never help — so the chance goes *up* from $\frac{2}{12}$ to $\frac{2}{11}$.

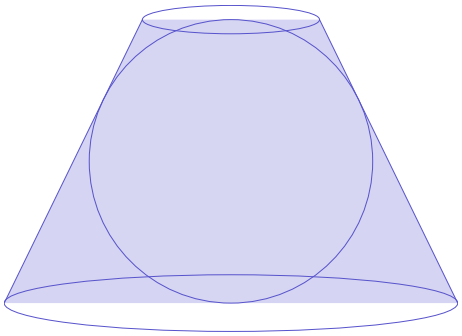
Watch out. Ignoring the relabelling and answering as for two normal dice gives $\frac{1}{6}$ — choice (C). Treating the two top numbers as independent draws from the bag (with replacement) gives the same $\frac{2}{12} = \frac{1}{6}$; but the two top tiles are always different tiles, so the draws are without replacement.

Answer: (D) $\frac{2}{11}$

10 Geometry

Source: 2014 AMC 10B #23

A sphere is inscribed in a truncated right circular cone as shown. The volume of the truncated cone is twice that of the sphere. What is the ratio of the radius of the bottom base of the truncated cone to the radius of the top base of the truncated cone?



(A) $\frac{3}{2}$ **(B)** $\frac{1+\sqrt{5}}{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $\frac{3+\sqrt{5}}{2}$

Key idea. Slice through the axis. The cross-section is an isosceles trapezoid with a circle inscribed in it, and *equal tangent segments* give the key relation: the sphere's radius squared equals the product of the two base radii. Then the volume condition becomes a quadratic.

Step 1: the cross-section. Cut the solid by a plane through its axis. We get an isosceles trapezoid with parallel sides $2r$ (top) and $2R$ (bottom), and a circle of radius a tangent to all four sides. The circle touches both bases, so the height is $h = 2a$.

Step 2: equal tangents. The circle touches the top base at its midpoint and the bottom base at its midpoint. Two tangent segments from the same point are equal, so from a top corner both tangents have length r, and from a bottom corner both have length R. Hence the slanted side has length $R + r$.

Step 3: Pythagoras. Drop a perpendicular from a top corner to the bottom base. It forms a right triangle with hypotenuse $R + r$, height $2a$, and horizontal leg $R - r$:

$$(R+r)^2 = (R-r)^2 + (2a)^2 \quad \Longrightarrow \quad 4Rr = 4a^2 \quad \Longrightarrow \quad a^2 = Rr.$$

Step 4: the volume condition. Scale so that $r = 1$; then $a^2 = R$, and R is the ratio we want. The truncated cone has volume

$$\frac{\pi h}{3}\left(R^2 + Rr + r^2\right) = \frac{2\pi a}{3}\left(R^2 + R + 1\right).$$

Setting this equal to twice the sphere's volume $\frac{4}{3}\pi a^3$:

$$\frac{2\pi a}{3}\left(R^2 + R + 1\right) = \frac{8\pi a^3}{3} \quad \Longrightarrow \quad R^2 + R + 1 = 4a^2 = 4R.$$

Step 5: solve. $R^2 - 3R + 1 = 0$ gives $R = \dfrac{3 \pm \sqrt{5}}{2}$. The bottom base is the larger one, so $R > 1$, and

$$R = \frac{3 + \sqrt{5}}{2} \approx 2.618.$$

Check: with $r = 1$ and $R = \frac{3+\sqrt{5}}{2}$ we get $a = \sqrt{R} = \frac{1+\sqrt{5}}{2}$, and the truncated cone's volume divided by the sphere's comes out to exactly 2. ✓

Watch out. If you set up the equation in terms of the sphere's radius (with $r = 1$), you get $a^4 - 3a^2 + 1 = 0$ and $a = \frac{1+\sqrt{5}}{2}$ — that is choice (B), but it is the sphere's radius, not the ratio R. The ratio is $R = a^2 = \frac{3+\sqrt{5}}{2}$.

Answer: (E) $\dfrac{3 + \sqrt{5}}{2}$