AMC 10

SET 24

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 13–23 of past AMC 10 contests (2000–2025) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**D**	2018 AMC 10B #14	Algebra
2	**C**	2023 AMC 10B #15	Number Theory
3	**D**	2013 AMC 10A #15	Geometry
4	**A**	2019 AMC 10B #16	Geometry
5	**D**	2005 AMC 10A #17	Counting & Probability
6	**D**	2025 AMC 10B #19	Geometry
7	**C**	2020 AMC 10A #21	Number Theory
8	**D**	2023 AMC 10A #21	Algebra
9	**B**	2019 AMC 10A #22	Counting & Probability
10	**C**	2015 AMC 10A #23	Algebra

1 Algebra

Source: 2018 AMC 10B #14

A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

(A) 202 **(B)** 223 **(C)** 224 **(D)** 225 **(E)** 234

Key idea. To use as few distinct values as possible, make every value repeat as often as it is allowed. The unique-mode condition caps every other value at 9 copies.

Step 1 (a bound). Suppose the list uses d distinct values. The mode appears 10 times. Every other value must appear fewer than 10 times (otherwise it would tie the mode), so at most 9 times. Hence

$$2018 \le 10 + 9(d-1).$$

This gives $9(d-1) \ge 2008$, so $d - 1 \ge \frac{2008}{9} = 223.1\ldots$. Since $d-1$ is a whole number, $d - 1 \ge 224$, so $d \ge 225$.

Step 2 (it can be done). Take one value 10 times, 223 other values 9 times each, and one more value once. That is

$$10 + 223 \cdot 9 + 1 = 10 + 2007 + 1 = 2018$$

numbers, with a unique mode occurring 10 times, using $1 + 223 + 1 = 225$ distinct values.

Both halves together show the least possible number is 225.

Watch out. The 224 from Step 1 counts only the values *other than* the mode. Forgetting to add the mode itself gives 224, choice (C).

Answer: (D) 225

2 Number Theory

Source: 2023 AMC 10B #15

What is the least positive integer m such that $m \cdot 2! \cdot 3! \cdot 4! \cdot 5! \cdots 16!$ is a perfect square?

(A) 30 **(B)** 30030 **(C)** 70 **(D)** 1430 **(E)** 1001

Key idea. A number is a perfect square exactly when every prime appears to an even power, so m must supply each prime that appears an odd number of times. Consecutive factorials share almost everything: $(2k+1)! = (2k)! \cdot (2k+1)$, so pairing them up throws away huge square factors at once.

Step 1: pair the factorials. For each k,

$$(2k)!\,(2k+1)! = \big((2k)!\big)^2\,(2k+1).$$

Group the product as $(2!\,3!)(4!\,5!)\cdots(14!\,15!)\cdot 16!$. Each pair is a perfect square times an odd number, so the whole product is a perfect square times

$$(3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15)\cdot 16!.$$

Step 2: the odd leftovers. $3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15 = 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13$. The primes with odd exponent are 7, 11, 13.

Step 3: the last factorial. Counting multiples, $16! = 2^{15}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13$ (for 2: $8+4+2+1=15$; for 3: $5+1=6$; for 5: 3; for 7: 2). The primes with odd exponent are 2, 5, 11, 13.

Step 4: combine. Multiplying, 11 and 13 now appear an even number of times, while 2, 5, 7 each still appear an odd number of times. So m must be divisible by $2\cdot 5\cdot 7 = 70$, and $m = 70$ makes every exponent even. The least m is 70.

Watch out. Stopping after the pairs and forgetting the unpaired 16! gives $7\cdot 11\cdot 13 = 1001$, choice (E). Looking only at 16! gives $2\cdot 5\cdot 11\cdot 13 = 1430$, choice (D).

Answer: (C) 70

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3 Geometry

Source: 2013 AMC 10A #15

Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 18

Key idea. Every side–altitude pair gives the same area, so (side)×(altitude) is the same number for all three sides. The statement talks only about altitudes, which is the cue to express each altitude through the area.

Step 1. Let the area be K. For a side s with altitude h, $K = \frac{1}{2}sh$, so $h = \frac{2K}{s}$. Hence

$$h_{10} = \frac{2K}{10} = \frac{K}{5}, \qquad h_{15} = \frac{2K}{15}.$$

Step 2. The altitude to the third side is their average. Writing $\frac{K}{5} = \frac{3K}{15}$,

$$h_3 = \frac{1}{2}\left(\frac{3K}{15} + \frac{2K}{15}\right) = \frac{1}{2}\cdot\frac{5K}{15} = \frac{K}{6}.$$

Step 3. If the third side is x, then $x\cdot h_3 = 2K$, so $x\cdot\frac{K}{6} = 2K$ and $x = 12$.

A triangle with sides 10, 15, 12 does exist ($10+12>15$), so the answer is consistent.

Another way. Altitudes are proportional to the *reciprocals* of the sides, so the condition says $\frac{1}{x}$ is the average of $\frac{1}{10}$ and $\frac{1}{15}$: $\frac{1}{x} = \frac{1}{2}\cdot\frac{5}{30} = \frac{1}{12}$. The third side is the harmonic mean of 10 and 15.

Watch out. Averaging the sides instead of the altitudes gives 12.5, which is not a choice. A longer side has a *shorter* altitude.

Answer: (D) 12

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4 Geometry

Source: 2019 AMC 10B #16

In $\triangle ABC$ with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC = CD$, $DE = EB$, and the ratio $AC : DE = 4 : 3$. What is the ratio $AD : DB$?

(A) $2 : 3$ **(B)** $2 : \sqrt{5}$ **(C)** $1 : 1$ **(D)** $3 : \sqrt{5}$ **(E)** $3 : 2$

Key idea. The conditions $AC = CD$ and $DE = EB$ create two isosceles triangles whose base angles are the two acute angles of the right triangle — and those add to $90°$. That forces a right angle at D, which turns the lengths into a Pythagorean calculation.

Only the ratio matters, so scale the figure so that $AC = CD = 4$ and $DE = EB = 3$. Write $\angle A$ and $\angle B$ for the acute angles of $\triangle ABC$; since $\angle C = 90°$, we have $\angle A + \angle B = 90°$.

Step 1. A right angle at D. Triangle ACD is isosceles with $CA = CD$, so its base angles match: $\angle CDA = \angle A$. Triangle DEB is isosceles with $ED = EB$, so $\angle EDB = \angle B$. Since D lies on segment AB, the three angles at D along the line add to $180°$:

$$\angle CDE = 180° - \angle A - \angle B = 180° - 90° = 90°.$$

Step 2. Find BC and AB. In right triangle CDE, $CE = \sqrt{4^2 + 3^2} = 5$. Then $BC = CE + EB = 5 + 3 = 8$, and in $\triangle ABC$,

$$AB = \sqrt{AC^2 + BC^2} = \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5}.$$

Step 3. Find AD. Drop the altitude CH from C to $\overline{AB}$. Because $\triangle ACD$ is isosceles, H is the midpoint of $\overline{AD}$. Right triangles AHC and ACB share $\angle A$, so they are similar and $\dfrac{AH}{AC} = \dfrac{AC}{AB}$:

$$AH = \frac{AC^2}{AB} = \frac{16}{4\sqrt{5}} = \frac{4}{\sqrt{5}}, \qquad AD = 2\,AH = \frac{8}{\sqrt{5}}.$$

Step 4. Find DB and the ratio.

$$DB = AB - AD = \frac{20}{\sqrt{5}} - \frac{8}{\sqrt{5}} = \frac{12}{\sqrt{5}}, \qquad AD : DB = 8 : 12 = 2 : 3.$$

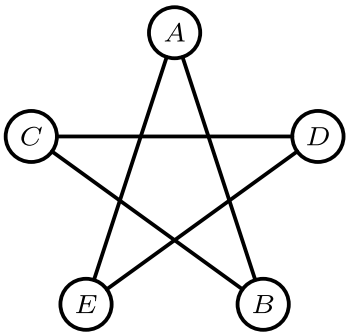
Check. With $C = (0,0)$, $A = (0,4)$, $B = (8,0)$, $E = (5,0)$, the point $D = \left(\frac{16}{5}, \frac{12}{5}\right)$ on $\overline{AB}$ has $CD = 4$ and $DE = 3$, as required.

Answer: (A) $2 : 3$

5 Counting & Probability

Source: 2005 AMC 10A #17

In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers 3, 5, 6, 7, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments AB, BC, CD, DE, and EA form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?



(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

Key idea. You do not need to find the arrangement. The middle term of a five-term arithmetic sequence is the average of the five terms, and the *total* of the five segment sums is fixed, because every letter is an endpoint of exactly two segments.

Step 1. Each letter appears in exactly two of the segments AB, BC, CD, DE, EA (for example, A is in AB and EA). So adding all five segment sums counts every number twice:

$$2(3+5+6+7+9) = 2 \cdot 30 = 60.$$

Step 2. Write the five terms of the arithmetic sequence as $m-2d,\ m-d,\ m,\ m+d,\ m+2d$. They add to $5m$, so $5m = 60$ and $m = 12$.

Check. Such an arrangement exists: $A=3$, $B=7$, $C=6$, $D=5$, $E=9$ gives segment sums $10, 13, 11, 14, 12$, which are $10, 11, 12, 13, 14$ in some order.

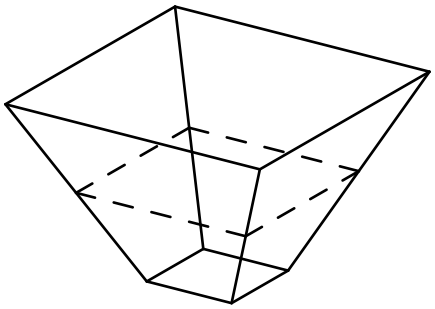
Watch out. Forgetting that each number is counted twice gives a total of 30 and an average of 6, which is not among the choices.

Answer: (D) 12

6 Geometry

Source: 2025 AMC 10B #19

A container has a 1×1 square bottom, a 3×3 open square top, and four congruent trapezoidal sides, as shown. Starting when the container is empty, a hose that runs water at a constant rate takes 35 minutes to fill the container up to the midline of the trapezoids.



How many more minutes will it take to fill the remainder of the container?

(A) 70 **(B)** 85 **(C)** 90 **(D)** 95 **(E)** 105

Key idea. The container is an upside-down square pyramid with its tip cut off. Extending the slanted sides down to the missing tip turns every volume question into "pyramid minus pyramid", and similar pyramids have volumes in the ratio of the *cubes* of their sizes.

Step 1. The side of a horizontal cross-section grows steadily from 1 at the bottom to 3 at the top, so at the midline it is $\frac{1+3}{2} = 2$.

Step 2. Extend the four slanted sides downward until they meet at a point. This makes three nested, similar pyramids (tip down) whose tops are squares of side 1, 2 and 3. Similar solids have volumes proportional to the cube of their size, so their volumes are in the ratio

$$1^3 : 2^3 : 3^3 = 1 : 8 : 27.$$

Call the volume of the smallest one 1 *part*.

Step 3. The water up to the midline is the side-2 pyramid minus the side-1 pyramid: $8 - 1 = 7$ parts. The rest of the container is $27 - 8 = 19$ parts.

Step 4. The first 7 parts take 35 minutes, so each part takes $35 \div 7 = 5$ minutes. The remaining 19 parts take $19 \cdot 5 = 95$ minutes.

Watch out. The midline is halfway *up*, but not halfway in volume — the container is much wider at the top. Assuming the remainder also takes 35 minutes gives an answer that is not even among the choices.

Answer: (D) 95

7 Number Theory

Source: 2020 AMC 10A #21

There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \cdots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}.$$

What is k?

(A) 117 **(B)** 136 **(C)** 137 **(D)** 273 **(E)** 306

Key idea. The exponents share a factor: $289 = 17 \cdot 17$. With $X = 2^{17}$ the fraction becomes $\frac{X^{17}+1}{X+1}$, which factors as an alternating polynomial. The question asks how many 1s the number has in binary, so rewrite the alternating sum as a sum of nonnegative blocks whose binary digits are easy to see.

Step 1: factor. Let $X = 2^{17}$, so $2^{289} = X^{17}$. Since 17 is odd, $X^{17}+1$ is divisible by $X+1$:

$$\frac{X^{17}+1}{X+1} = X^{16} - X^{15} + X^{14} - X^{13} + \cdots + X^2 - X + 1.$$

(Multiply out $(X+1)(X^{16} - X^{15} + \cdots - X + 1)$: the middle terms cancel in pairs, leaving $X^{17}+1$.)

Step 2: group in pairs. The 16 terms before the final $+1$ form 8 pairs:

$$(X^{16} - X^{15}) + (X^{14} - X^{13}) + \cdots + (X^2 - X) + 1.$$

A typical pair is $X^{2j} - X^{2j-1} = X^{2j-1}(X-1) = 2^{17(2j-1)}\,(2^{17}-1)$ for $j = 1, 2, \ldots, 8$.

Step 3: read off binary digits. The number $2^{17}-1$ is 17 ones in binary. Multiplying by $2^{17(2j-1)}$ shifts it, so pair j is a block of 17 ones in positions $34j-17$ through $34j-1$. The blocks for consecutive j are separated by a gap (block j ends at $34j-1$; block $j+1$ starts at $34j+17$), and the final $+1$ sits in position 0, below the first block at positions 17–33. Nothing overlaps, so there are no carries.

Step 4. The binary expansion therefore has $8 \cdot 17 + 1 = 137$ ones, and k is exactly that number of ones: $k = 137$.

Watch out. Forgetting the final $+1$ gives $8 \cdot 17 = 136$, choice (B).

Answer: (C) 137

8 Algebra

Source: 2023 AMC 10A #21

There is a unique polynomial $P(x)$ of least degree with leading coefficient 1 satisfying all of the following:

1 is a root of $P(x)-1$, 2 is a root of $P(x-2)$, 3 is a root of $P(3x)$, and 4 is a root of $4P(x)$.

All the roots of $P(x)$ except one are integers. If the one non-integer root can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, what is $m+n$?

(A) 41 **(B)** 43 **(C)** 45 **(D)** 47 **(E)** 49

Key idea. The four conditions are disguised statements about values of P. Substitute each number into its expression to turn every condition into "P(something) = value"; then three of them say where P has roots, and the fourth pins down whatever is left.

Step 1: decode the conditions.

- 1 is a root of $P(x)-1$: $P(1)-1=0$, so $P(1)=1$.
- 2 is a root of $P(x-2)$: $P(2-2)=0$, so $P(0)=0$.
- 3 is a root of $P(3x)$: $P(3\cdot 3)=0$, so $P(9)=0$.
- 4 is a root of $4P(x)$: $4P(4)=0$, so $P(4)=0$.

Step 2: the degree. P has the three different roots 0, 4, 9, so its degree is at least 3. The only monic cubic with these roots is $x(x-4)(x-9)$, and at $x=1$ it equals

$$1\cdot(-3)\cdot(-8)=24\neq 1.$$

So no cubic works, and the degree is at least 4.

Step 3: the quartic. A monic quartic with roots $0, 4, 9$ has the form

$$P(x)=x(x-4)(x-9)(x-c)$$

for some number c. The last condition gives

$$P(1)=24(1-c)=1, \quad \text{so} \quad 1-c=\frac{1}{24}, \quad c=\frac{23}{24}.$$

This c is the only value that works, so P is unique, as promised.

Step 4: read off the answer. The roots are 0, 4, 9 and $\frac{23}{24}$; the only non-integer one is $\frac{23}{24}$, already in lowest terms. So $m+n=23+24=47$.

Watch out. Reading "2 is a root of $P(x-2)$" as $P(2)=0$ gives roots $2, 4, 9$. Then $P(1)=(-1)(-3)(-8)(1-c)=-24(1-c)=1$, so $c=\frac{25}{24}$ and $m+n=49$, choice (E). Always substitute the number into the *whole* expression: $x=2$ makes $x-2=0$.

Answer: (D) 47

9 Counting & Probability

Source: 2019 AMC 10A #22

Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads, and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0,1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x-y| > \frac{1}{2}$?

(A) $\frac{1}{3}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{2}{3}$

Key idea. Each number is one of three kinds: exactly 0, exactly 1, or uniform on $[0,1]$. Split into cases by the kinds of x and y; the only case with two continuous numbers is a geometric-probability picture in the unit square.

Step 1. The kinds. A number is 0 with probability $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$, is 1 with probability $\frac{1}{4}$, and is uniform on $[0,1]$ with probability $\frac{1}{2}$. So a number is "discrete" (0 or 1) with probability $\frac{1}{2}$, and each of the four combinations below has probability $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$.

Case 1: both discrete. Each of x, y is 0 or 1, equally likely. Then $|x-y|$ is 0 or 1, and $|x-y| > \frac{1}{2}$ exactly when $x \neq y$: probability $\frac{1}{2}$.

Case 2: x discrete, y uniform. If $x=0$ we need $y > \frac{1}{2}$; if $x=1$ we need $y < \frac{1}{2}$. Either way the chance is $\frac{1}{2}$.

Case 3: x uniform, y discrete. By symmetry with Case 2, the chance is $\frac{1}{2}$.

Case 4: both uniform. The point (x,y) is uniform in the unit square. The region $|x-y| > \frac{1}{2}$ is two corner triangles, near $(1,0)$ and $(0,1)$, each with legs $\frac{1}{2}$, so their total area is $2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$. The chance is $\frac{1}{4}$.

Combine. Each case has weight $\frac{1}{4}$:

$$\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}\right)=\frac{1}{4}\cdot\frac{7}{4}=\frac{7}{16}.$$

Watch out. In Case 1 it is tempting to think "0 and 1 are far apart, so this case always works." But x and y can both be 0 or both be 1. Counting Case 1 as probability 1 gives $\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}\right)=\frac{9}{16}$, choice (D).

Answer: (B) $\frac{7}{16}$

10 Algebra

Source: 2015 AMC 10A #23

The zeros of the function

$$f(x) = x^2 - ax + 2a$$

are integers. What is the sum of the possible values of a?

(A) 7 **(B)** 8 **(C)** 16 **(D)** 17 **(E)** 18

Key idea. "The zeros are integers" invites naming them r and s and using Vieta's formulas. The two resulting equations combine into $rs = 2(r+s)$ — a product equal to a multiple of a sum, which is exactly the shape that factors after adding a constant ("Simon's Favourite Factoring Trick").

Step 1. Vieta. If the zeros are the integers r and s, then $x^2 - ax + 2a = (x - r)(x - s)$, so

$$r + s = a, \qquad rs = 2a.$$

Eliminating a gives $rs = 2(r + s)$.

Step 2. Factor. Move everything to one side and add 4:

$$rs - 2r - 2s = 0 \implies rs - 2r - 2s + 4 = 4 \implies (r - 2)(s - 2) = 4.$$

Step 3. List the factor pairs. The integers $r - 2$ and $s - 2$ multiply to 4. Ignoring order (swapping r and s gives the same a), the pairs are

$$(1, 4), \quad (2, 2), \quad (-1, -4), \quad (-2, -2).$$

Since $a = r + s = (r - 2) + (s - 2) + 4$, these give

$$a = 1 + 4 + 4 = 9, \quad a = 2 + 2 + 4 = 8, \quad a = -1 - 4 + 4 = -1, \quad a = -2 - 2 + 4 = 0.$$

Step 4. Check each one.

- $a = 9$: $x^2 - 9x + 18 = (x - 3)(x - 6)$.
- $a = 8$: $x^2 - 8x + 16 = (x - 4)^2$.
- $a = -1$: $x^2 + x - 2 = (x + 2)(x - 1)$.
- $a = 0$: x^2, with the double zero 0.

All four work, and Step 3 shows there are no others. The sum of the possible values is

$$9 + 8 + (-1) + 0 = 16.$$

Watch out. Two cases are easy to lose. Skipping the negative factor pairs leaves $9 + 8 = 17$, choice (D). Skipping the equal pairs $(2, 2)$ and $(-2, -2)$ (double zeros) leaves $9 + (-1) = 8$, choice (B).

Answer: (C) 16