AMC 10

SET 25

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 13–23 of past AMC 10 contests (2000–2025) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**E**	2013 AMC 10B #14	Algebra
2	**C**	2006 AMC 10B #15	Geometry
3	**D**	2022 AMC 10B #15	Algebra
4	**B**	2020 AMC 10A #16	Counting & Probability
5	**B**	2008 AMC 10A #17	Geometry
6	**C**	2017 AMC 10A #19	Counting & Probability
7	**D**	2013 AMC 10A #21	Number Theory
8	**E**	2021 Fall AMC 10A #21	Counting & Probability
9	**B**	2024 AMC 10B #23	Algebra
10	**C**	2019 AMC 10B #23	Geometry

1 Algebra

Source: 2013 AMC 10B #14

Define $a\clubsuit b = a^2b - ab^2$. Which of the following describes the set of points (x, y) for which $x\clubsuit y = y\clubsuit x$?

(A) A finite set of points **(B)** One line **(C)** Two parallel lines

(D) Two intersecting lines **(E)** Three lines

Key idea. Write out both sides, move everything to one side, and factor. A product is zero exactly when one of its factors is zero, and here each factor turns out to be linear — so each one is a line.

Step 1. By the definition, $x\clubsuit y = x^2y - xy^2$. Swapping the roles of the two inputs, $y\clubsuit x = y^2x - yx^2$, which is $xy^2 - x^2y$.

Step 2. Setting them equal:

$$x^2y - xy^2 = xy^2 - x^2y.$$

Moving everything to the left gives $2x^2y - 2xy^2 = 0$, and factoring gives

$$2xy(x - y) = 0.$$

Step 3. This holds exactly when $x = 0$, or $y = 0$, or $x = y$. These are the y-axis, the x-axis, and the line $y = x$: three different lines (all passing through the origin). Every point on any of them works, and no other point does.

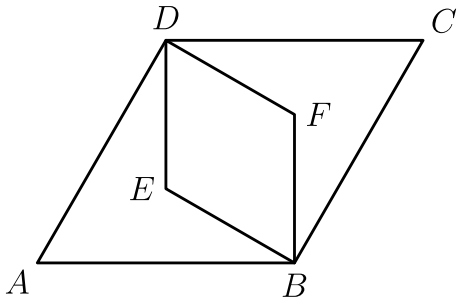
Watch out. Do not divide both sides by xy. That cancels the factors x and y and leaves only $x = y$, so you would pick "One line" — choice (B) — and lose the two axes.

Answer: (E) Three lines

2 Geometry

Source: 2006 AMC 10B #15

Rhombus $ABCD$ is similar to rhombus $BFDE$. The area of rhombus $ABCD$ is 24, and $\angle BAD = 60°$. What is the area of rhombus $BFDE$?



(A) 6 **(B)** $4\sqrt{3}$ **(C)** 8 **(D)** 9 **(E)** $6\sqrt{3}$

Key idea. A 60° rhombus is two equilateral triangles glued along the short diagonal BD. The key is to notice that BD is the *short* diagonal of $ABCD$ but the *long* diagonal of $BFDE$ (in the figure $BFDE$ is pointed at B and D).

Step 1. Since $AB = AD$ and $\angle BAD = 60^\circ$, triangle ABD is equilateral, and so is triangle CBD. Each has area $\frac{24}{2} = 12$.

Step 2. In a rhombus with a 60° angle, the long diagonal is $\sqrt{3}$ times the short one. (Its half-diagonals are the legs of a 30°–60°–90° triangle, in ratio $1 : \sqrt{3}$.) Let the side of $ABCD$ be s, so $BD = s$.

Step 3. In $BFDE$ the angles at B and D are the 60° angles, so BD is its long diagonal. If its short diagonal is t, then $s = \sqrt{3}\,t$. So the short diagonals are in ratio $t : s = 1 : \sqrt{3}$, and $BFDE$ is $ABCD$ scaled by $\frac{1}{\sqrt{3}}$.

Step 4. Areas scale by the square of the ratio: $24 \cdot \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{24}{3} = 8$.

Another way. In rhombus $BFDE$ we have $EB = ED$ and $\angle EBD = 30^\circ$ (half of its 60° angle at B), so E is the center of equilateral triangle ABD, and segments EA, EB, ED cut that triangle into three congruent triangles of area $\frac{12}{3} = 4$. Likewise F is the center of CBD. Rhombus $BFDE$ is triangle BED plus triangle BFD, so its area is $4 + 4 = 8$.

Watch out. Squaring is essential: scaling the area by $\frac{1}{\sqrt{3}}$ instead of $\frac{1}{3}$ gives $8\sqrt{3}$, which is not a choice.

Answer: (C) 8

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3 Algebra

Source: 2022 AMC 10B #15

Let S_n be the sum of the first n terms of an arithmetic sequence that has a common difference of 2. The quotient $\dfrac{S_{3n}}{S_n}$ does not depend on n. What is S_{20}?

(A) 340 **(B)** 360 **(C)** 380 **(D)** 400 **(E)** 420

Key idea. Write S_n as a formula in n and the unknown first term a. "Does not depend on n" means the quotient takes the same value for $n = 1$ and $n = 2$; that single equation pins down a.

Step 1. The terms are $a, a+2, a+4, \ldots$, so the nth term is $a+2(n-1)$. The sum is (number of terms) $\times$ (average of first and last):

$$S_n = n \cdot \frac{a + \big(a + 2(n-1)\big)}{2} = n(a + n - 1).$$

Step 2. Then

$$\frac{S_{3n}}{S_n} = \frac{3n(a + 3n - 1)}{n(a + n - 1)} = \frac{3(a + 3n - 1)}{a + n - 1}.$$

Step 3. The value at $n = 1$ must equal the value at $n = 2$:

$$\frac{3(a+2)}{a} = \frac{3(a+5)}{a+1}.$$

Cross-multiplying, $(a+2)(a+1) = a(a+5)$, so $a^2 + 3a + 2 = a^2 + 5a$, giving $a = 1$.

Step 4. Check: with $a = 1$ the sequence is $1, 3, 5, \ldots$ and $S_n = n^2$, so $\frac{S_{3n}}{S_n} = \frac{9n^2}{n^2} = 9$ for every n. Hence $S_{20} = 20^2 = 400$.

Watch out. Guessing the even numbers $2, 4, 6, \ldots$ (because the difference is 2) gives $S_{20} = 420$, choice (E). But then $S_n = n(n+1)$ and $\frac{S_{3n}}{S_n} = \frac{3(3n+1)}{n+1}$, which is 6 at $n = 1$ and 7 at $n = 2$ — it does depend on n.

Answer: (D) 400

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4 Counting & Probability

Source: 2020 AMC 10A #16

A point is chosen at random within the square in the coordinate plane whose vertices are $(0, 0)$, $(2020, 0)$, $(2020, 2020)$, and $(0, 2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x, y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

(A) 0.3 **(B)** 0.4 **(C)** 0.5 **(D)** 0.6 **(E)** 0.7

Key idea. The huge square is made of 2020^2 identical unit squares, each with lattice points at its four corners. So the probability over the whole square equals the probability inside a single unit square — and there the "close" region is four quarter-circles.

Step 1. Because 2020 is an integer, the square is tiled exactly by 2020×2020 unit squares, and every unit square looks the same: lattice points at its four corners and none inside. So the probability is the fraction of *one* unit square lying within d of a corner.

Step 2. Suppose $d < \frac{1}{2}$. Around each corner the close points form a quarter-circle of radius d, and the four quarter-circles do not overlap (two adjacent corners are 1 apart, more than $2d$). Their total area is $4 \cdot \frac{1}{4}\pi d^2 = \pi d^2$, and the unit square has area 1, so the probability is πd^2.

Step 3. Set $\pi d^2 = \frac{1}{2}$: then $d^2 = \dfrac{1}{2\pi}$ and

$$d = \sqrt{\frac{1}{2\pi}} \approx \sqrt{\frac{1}{6.283}} \approx \sqrt{0.159} \approx 0.399.$$

This is indeed less than $\frac{1}{2}$, so Step 2 applies. (If d were $\frac{1}{2}$ or more, the probability would already be at least $\pi \cdot \frac{1}{4} \approx 0.785 > \frac{1}{2}$, so $d < \frac{1}{2}$ is forced.)

Step 4. Since $0.35 < 0.399 < 0.45$, d to the nearest tenth is 0.4.

Watch out. Each unit square touches four lattice points, not one. Using a single quarter-circle, $\frac{1}{4}\pi d^2 = \frac{1}{2}$, gives $d \approx 0.80$, which is not a choice.

Answer: (B) 0.4

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5 Geometry

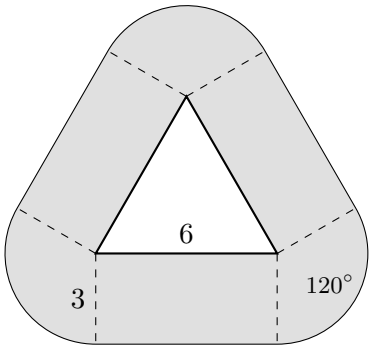
Source: 2008 AMC 10A #17

An equilateral triangle has side length 6. What is the area of the region containing all points that are outside the triangle and not more than 3 units from a point of the triangle?

(A) $36 + 24\sqrt{3}$ **(B)** $54 + 9\pi$ **(C)** $54 + 18\sqrt{3} + 6\pi$

(D) $\left(2\sqrt{3} + 3\right)^2 \pi$ **(E)** $9\left(\sqrt{3} + 1\right)^2 \pi$

Key idea. Walk once around the outside of the triangle at distance 3. Along each side the region is a rectangle; at each corner it is a slice of a circle. Add the pieces.



Step 1. Points within 3 of a side (and outside the triangle) fill a 6×3 rectangle on that side. Three rectangles give $3 \cdot 6 \cdot 3 = 54$.

Step 2. Near a vertex, the points are within 3 of the vertex itself: a sector of radius 3. Around the vertex the full $360°$ is split into the triangle's $60°$, two right angles from the rectangles, and the sector. So the sector angle is $360° - 60° - 90° - 90° = 120°$.

Step 3. Three $120°$ sectors make one full circle of radius 3, with area $\pi \cdot 3^2 = 9\pi$. The total area is $54 + 9\pi$.

Watch out. Taking each corner piece to be a $60°$ sector (the triangle's angle) instead of $120°$. The corner gap is what is left *outside* the triangle and the two rectangles.

Answer: (B) $54 + 9\pi$

6 Counting & Probability

Source: 2017 AMC 10A #19

> Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?
>
> **(A)** 12 **(B)** 16 **(C)** 28 **(D)** 32 **(E)** 40

Key idea. Alice has the most restrictions, so place her first. Whether she sits on an end or in the middle decides how many neighbors she has, so split into those two cases.

Case 1: Alice in the middle (seat 2, 3, or 4). She has two neighbors, and neither may be Bob or Carla, so her neighbors are Derek and Eric. That keeps Derek and Eric apart automatically (Alice is between them). Bob and Carla fill the remaining two seats. Count: 3 seats for Alice, 2 ways to put Derek and Eric on her two sides, 2 ways to place Bob and Carla — that is $3 \cdot 2 \cdot 2 = 12$.

Case 2: Alice on an end (2 choices). Her only neighbor must be Derek or Eric (2 choices); call that person X. The person next to X cannot be the other of Derek/Eric, so must be Bob or Carla (2 choices). The last two seats get the remaining two people in either order (2 ways), and now Derek and Eric are separated by at least that Bob-or-Carla. Count: $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

Total. $12 + 16 = 28$. (A computer check over all 120 seatings confirms 28.)

Watch out. Counting only one of the two cases gives 16 (B) or 12 (A). Alice's position really does change the count, so both cases are needed.

Answer: (C) 28

7 Number Theory

Source: 2013 AMC 10A #21

> A group of 12 pirates agree to divide a treasure chest of gold coins among themselves as follows. The k^{th} pirate to take a share takes $\frac{k}{12}$ of the coins that remain in the chest. The number of coins initially in the chest is the smallest number for which this arrangement will allow each pirate to receive a positive whole number of coins. How many coins does the 12^{th} pirate receive?
>
> **(A)** 720 **(B)** 1296 **(C)** 1728 **(D)** 1925 **(E)** 3850

Key idea. The 12^{th} pirate takes $\frac{12}{12}$ — everything that is left. So call his share n and *work backward*: each earlier step multiplies the chest by a known fraction, and the whole question becomes "which n makes all those numbers integers?"

Step 1. Let R_k be the number of coins in the chest just before pirate k takes his share. Pirate k takes $\frac{k}{12}R_k$, leaving

$$R_{k+1} = \frac{12-k}{12}\,R_k, \qquad \text{so} \qquad R_k = \frac{12}{12-k}\,R_{k+1}.$$

Step 2. Pirate 12 takes all of R_{12}, so $R_{12} = n$ is his share. Going back to the start,

$$R_1 = n \cdot \frac{12}{11} \cdot \frac{12}{10} \cdots \frac{12}{1} = n \cdot \frac{12^{11}}{11!}.$$

Step 3. Factor: $12^{11} = 2^{22} \cdot 3^{11}$ and $11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$. Hence

$$R_1 = n \cdot \frac{2^{14} \cdot 3^7}{5^2 \cdot 7 \cdot 11} = n \cdot \frac{2^{14} \cdot 3^7}{1925}.$$

The top shares no prime with 1925, so R_1 is a whole number only if 1925 divides n. Thus $n \geq 1925$. Since R_1 grows with n, the smallest starting chest comes from the smallest n that works.

Step 4. Check that $n = 1925$ really works. Writing $m = 12 - k$, the chest before pirate k holds

$$R_k = 1925 \cdot \frac{12^m}{m!} \qquad (1 \leq m \leq 11).$$

The factors 5, 7, 11 of $m!$ all divide 1925, because they divide 11!. The power of 2 in $m!$ is less than m and the power of 3 is less than $\frac{m}{2}$, while $12^m = 2^{2m}3^m$ supplies plenty of both. So every R_k is an integer.

Step 5. Each share is $R_k - R_{k+1}$, a difference of integers, and it is positive because pirate k takes $\frac{k}{12} > 0$ of a positive pile. For example, the chest starts with $R_1 = 2^{14} \cdot 3^7 = 35{,}831{,}808$ coins, pirate 1 takes $\frac{1}{12}$ of it, 2,985,984, and the last few shares are 21,175 (pirate 11) and 1925 (pirate 12).

So the 12^{th} pirate receives $n = 1925$ coins.

Answer: (D) 1925

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8 Counting & Probability

Source: 2021 Fall AMC 10A #21

> Each of 20 balls is tossed independently and at random into one of the 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?
>
> **(A)** 1 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 16

Key idea. Both p and q are "favourable outcomes over all outcomes" with the *same* total, 5^{20}. So $\frac{p}{q}$ is just a ratio of two counts, and the huge numbers like 20! will cancel. That is why the problem asks for the ratio rather than for p or q.

Step 1. The sample space. Think of the balls as numbered 1 to 20 and the bins as labelled. Each ball has 5 choices, so there are 5^{20} equally likely outcomes. Hence

$$\frac{p}{q} = \frac{\#\{\text{outcomes with counts } 3, 5, 4, 4, 4 \text{ in some order}\}}{\#\{\text{outcomes with every bin holding } 4\}}.$$

Step 2. Counting for q. Choose which 4 balls go in bin 1, then 4 of the remaining 16 for bin 2, and so on:

$$\binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4} = \frac{20!}{4!\,4!\,4!\,4!\,4!}.$$

Step 3. Counting for p. First decide which bin gets 3 balls (5 ways) and which gets 5 balls (4 ways); these roles are different, so order matters and there are $5 \cdot 4 = 20$ choices. For each choice, the same reasoning as in Step 2 gives $\dfrac{20!}{3!\,5!\,4!\,4!\,4!}$ ways to place the balls. So the count is

$$20 \cdot \frac{20!}{3!\,5!\,4!\,4!\,4!}.$$

Step 4. Divide. The factor 20! and three factors of 4! cancel:

$$\frac{p}{q} = 20 \cdot \frac{4! \cdot 4!}{3! \cdot 5!} = 20 \cdot \frac{4!}{3!} \cdot \frac{4!}{5!} = 20 \cdot 4 \cdot \frac{1}{5} = 16.$$

A useful way to see the $\frac{4}{5}$: turning a $(4, 4)$ pair of bins into a $(3, 5)$ pair changes the count by the factor $\frac{4!\,4!}{3!\,5!} = \frac{24 \cdot 24}{6 \cdot 120} = \frac{4}{5}$, slightly *fewer* ways for one specific pair of bins. But there are 20 ordered choices of which bins play those roles, so overall p is 16 times q.

Watch out. Choosing the two special bins with $\binom{5}{2} = 10$ instead of $5 \cdot 4 = 20$ gives $10 \cdot \frac{4}{5} = 8$, choice (C). The "3-ball bin" and the "5-ball bin" are different jobs, so swapping them gives a different outcome.

Answer: (E) 16

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9 Algebra

Source: 2024 AMC 10B #23

The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. What is

$$\frac{F_2}{F_1} + \frac{F_4}{F_2} + \frac{F_6}{F_3} + \cdots + \frac{F_{20}}{F_{10}}?$$

(A) 318 **(B)** 319 **(C)** 320 **(D)** 321 **(E)** 322

Key idea. The terms are F_{2k}/F_k for $k = 1, \ldots, 10$. Ten divisions is not many, so compute the first few — the quotients turn out to be whole numbers following a Fibonacci-like rule, and that pattern finishes the job.

Step 1: the Fibonacci numbers. $F_1, \ldots, F_{20}$ are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765.$$

Step 2: the quotients.

$$\tfrac{F_2}{F_1} = 1,\ \tfrac{F_4}{F_2} = 3,\ \tfrac{F_6}{F_3} = \tfrac{8}{2} = 4,\ \tfrac{F_8}{F_4} = \tfrac{21}{3} = 7,\ \tfrac{F_{10}}{F_5} = \tfrac{55}{5} = 11.$$

Each is the sum of the previous two ($1 + 3 = 4$, $3 + 4 = 7$, $4 + 7 = 11$). These are the *Lucas numbers* L_k, defined by $L_1 = 1$, $L_2 = 3$, $L_k = L_{k-1} + L_{k-2}$. The reason is the identity $F_{2k} = F_k\,(F_{k-1} + F_{k+1})$

(with $F_0 = 0$): the quotient is $F_{k-1} + F_{k+1}$, a sum of two Fibonacci sequences, so it obeys the same rule. Continuing, $L_6, \ldots, L_{10} = 18, 29, 47, 76, 123$; for instance $\frac{F_{20}}{F_{10}} = \frac{6765}{55} = 123$.

Step 3: the sum. Add directly:

$$1 + 3 + 4 + 7 + 11 + 18 + 29 + 47 + 76 + 123 = 319.$$

Or telescope: $L_k = L_{k+2} - L_{k+1}$, so $\sum_{k=1}^{10} L_k = L_{12} - L_2 = 322 - 3 = 319$, using $L_{11} = 76 + 123 = 199$ and $L_{12} = 123 + 199 = 322$.

Watch out. Borrowing the Fibonacci fact $F_1 + \cdots + F_n = F_{n+2} - 1$ and writing $L_{12} - 1 = 321$ gives choice (D). The telescoping subtracts $L_2 = 3$, not 1.

Answer: (B) 319

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10 Geometry

Source: 2019 AMC 10B #23

Points $A = (6, 13)$ and $B = (12, 11)$ lie on a circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

(A) $\frac{83\pi}{8}$ **(B)** $\frac{21\pi}{2}$ **(C)** $\frac{85\pi}{8}$ **(D)** $\frac{43\pi}{4}$ **(E)** $\frac{87\pi}{8}$

Key idea. Two tangents from one point P are equally long, so P is on the perpendicular bisector of $\overline{AB}$ — and so is the centre. The phrase "intersect at a point on the x-axis" is what pins P down; then "radius $\perp$ tangent" finds the centre.

Step 1: find P. Let P be where the tangents meet. Tangent segments from a point are equal, so $PA = PB$, and P lies on the perpendicular bisector of $\overline{AB}$.

The midpoint of $\overline{AB}$ is $M = (9, 12)$. The slope of $\overline{AB}$ is $\frac{11-13}{12-6} = -\frac{1}{3}$, so the perpendicular bisector has slope 3:

$$y - 12 = 3(x - 9), \quad \text{i.e.} \quad y = 3x - 15.$$

Setting $y = 0$ gives $P = (5, 0)$.

Step 2: find the centre O. The centre is equidistant from A and B, so O is also on $y = 3x - 15$. The tangent PA has slope $\frac{13-0}{6-5} = 13$, and the radius OA is perpendicular to it, so OA has slope $-\frac{1}{13}$:

$$y - 13 = -\tfrac{1}{13}(x - 6).$$

Substitute $y = 3x - 15$: $3x - 28 = -\frac{x-6}{13}$, so $39x - 364 = -x + 6$, $40x = 370$, $x = \frac{37}{4}$, and $y = \frac{111}{4} - \frac{60}{4} = \frac{51}{4}$. So $O = \left(\frac{37}{4}, \frac{51}{4}\right)$.

Step 3: the radius.

$$r^2 = OA^2 = \left(\tfrac{37}{4} - 6\right)^2 + \left(\tfrac{51}{4} - 13\right)^2 = \left(\tfrac{13}{4}\right)^2 + \left(-\tfrac{1}{4}\right)^2 = \frac{170}{16} = \frac{85}{8}.$$

(The same value comes out for OB^2: $\left(-\frac{11}{4}\right)^2 + \left(\frac{7}{4}\right)^2 = \frac{170}{16}$.) The area is $\pi r^2 = \frac{85\pi}{8}$.

Another way. Triangle PAO has a right angle at A, and $\overline{AM}$ is its altitude to the hypotenuse $\overline{PO}$ (since $\overline{AB} \perp \overline{PO}$). Triangles OAP and AMP are similar, so $\frac{OA}{PA} = \frac{AM}{PM}$. With $PA^2 = 1 + 169 = 170$, $AM^2 = 9 + 1 = 10$, $PM^2 = 16 + 144 = 160$:

$$r^2 = PA^2 \cdot \frac{AM^2}{PM^2} = 170 \cdot \frac{10}{160} = \frac{85}{8}.$$

Watch out. The midpoint M of $\overline{AB}$ is *not* the centre — $\overline{AB}$ is a chord, not a diameter. Using M gives $r^2 = AM^2 = 10$ and area 10π, which is not a choice.

Answer: (C) $\dfrac{85\pi}{8}$