AMC 10

SET 26

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 13–23 of past AMC 10 contests (2000–2025) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**D**	2021 Fall AMC 10A #13	Counting & Probability
2	**D**	2024 AMC 10A #14	Geometry
3	**D**	2005 AMC 10B #16	Algebra
4	**A**	2024 AMC 10B #16	Counting & Probability
5	**D**	2007 AMC 10B #17	Geometry
6	**B**	2025 AMC 10A #18	Algebra
7	**B**	2022 AMC 10A #21	Geometry
8	**D**	2011 AMC 10A #21	Counting & Probability
9	**B**	2018 AMC 10B #23	Number Theory
10	**B**	2023 AMC 10B #23	Algebra

1 Counting & Probability

Source: 2021 Fall AMC 10A #13

Each of 6 balls is randomly and independently painted either black or white with equal probability. What is the probability that every ball is different in color from more than half of the other 5 balls?

(A) $\frac{1}{64}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{5}{16}$ **(E)** $\frac{1}{2}$

Key idea. Translate the condition into a statement about how many balls have each color. "More than half of 5" means "at least 3".

Step 1. Suppose a ball's color is shared by k of the 6 balls (counting itself). Then it differs in color from $6-k$ of the other 5 balls. The condition says $6-k \geq 3$, i.e. $k \leq 3$.

Step 2. This must hold for a black ball and for a white ball. If there are b black and w white balls with $b+w=6$, we need $b \leq 3$ and $w \leq 3$, which forces $b=w=3$. (If all balls were one color, every ball would differ from 0 others, so that fails too.) Conversely, with 3 of each color, every ball differs from exactly 3 of the other 5.

Step 3. There are $2^6 = 64$ equally likely colorings, and $\binom{6}{3} = 20$ ways to choose which 3 balls are white. The probability is

$$\frac{20}{64} = \frac{5}{16}.$$

Answer: (D) $\frac{5}{16}$

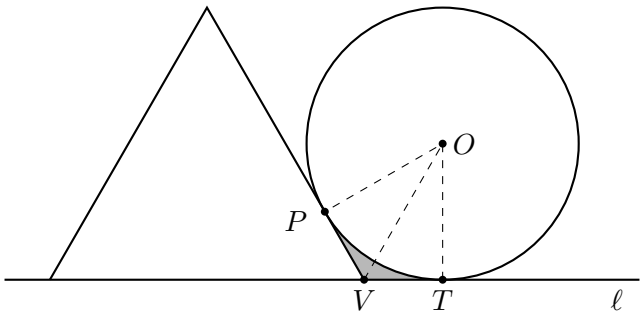
2 Geometry

Source: 2024 AMC 10A #14

One side of an equilateral triangle of height 24 lies on line ℓ. A circle of radius 12 is tangent to ℓ and is externally tangent to the triangle. The area of the region exterior to the triangle and the circle and bounded by the triangle, the circle, and line ℓ can be written as $a\sqrt{b}-c\pi$, where a, b, and c are positive integers and b is not divisible by the square of any prime. What is $a+b+c$?

(A) 72 **(B)** 73 **(C)** 74 **(D)** 75 **(E)** 76

Key idea. The region is a curved corner: it is cut off by two tangent lines from one point V and the circle between them. Whenever two tangents come from one point, draw the radii to the tangency points — you get a kite, and the region is "kite minus sector."



Step 1: the angle at V. Let V be the base vertex of the triangle next to the circle, O the center, T the tangency point on ℓ, and P the tangency point on the side of the triangle. The triangle's angle at V is 60°, so the angle on the outside, between ℓ and the side VP, is $180^\circ - 60^\circ = 120^\circ$. The circle sits inside this 120° angle.

Step 2: the kite. Radii to tangency points are perpendicular to the tangents, so $\angle VTO = \angle VPO = 90^\circ$, and the two tangent lengths are equal, $VT = VP$. Segment VO bisects the 120° angle, so in right triangle VTO, $\angle TVO = 60^\circ$ and

$$VT = \frac{OT}{\tan 60^\circ} = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$$

Kite $VTOP$ is two copies of triangle VTO, so its area is $2 \cdot \frac{1}{2} \cdot 4\sqrt{3} \cdot 12 = 48\sqrt{3}$.

Step 3: the sector. The angles of the kite add to 360°, so $\angle TOP = 360^\circ - 90^\circ - 90^\circ - 120^\circ = 60^\circ$. The sector of the circle inside the kite has area $\frac{60}{360} \cdot \pi \cdot 12^2 = 24\pi$.

Step 4. The shaded region is the kite minus that sector: $48\sqrt{3} - 24\pi$. Here $b = 3$ is squarefree, so $a = 48$, $b = 3$, $c = 24$, and $a + b + c = 75$.

(Notice that the height 24 was never used: the region depends only on the 120° angle and the radius. The tangency point P does land on the side of the triangle, which is $16\sqrt{3}$ long, since $VP = 4\sqrt{3}$.)

Watch out. The kite's angle at V is the *outside* angle 120°, not the triangle's 60°. Using 60° there makes the sector 120° and the tangent length $12\sqrt{3}$, and the result matches no choice.

Answer: (D) 75

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3 Algebra

Source: 2005 AMC 10B #16

> The quadratic equation $x^2+mx+n=0$ has roots that are twice those of $x^2+px+m=0$, and none of m, n, and p is zero. What is the value of $\frac{n}{p}$?
>
> **(A)** 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

Key idea. The problem talks about *roots* but asks about *coefficients*, and the bridge between them is Vieta's formulas: for $x^2+bx+c=0$ the roots add to $-b$ and multiply to c. Name the roots of $x^2+px+m=0$ and write both equations' coefficients in terms of them.

Step 1. Let r and s be the roots of $x^2+px+m=0$. By Vieta,

$$r+s=-p, \qquad rs=m.$$

Step 2. The roots of $x^2+mx+n=0$ are $2r$ and $2s$. By Vieta again,

$$2r+2s=-m, \qquad (2r)(2s)=n.$$

So $m=-2(r+s)=2p$ and $n=4rs=4m$.

Step 3. Combine: $n=4m=4\cdot 2p=8p$. Since $p\neq 0$ we may divide, and $\dfrac{n}{p}=8$.

Check: take $p=-1$, so $m=-2$ and $n=-8$. Then $x^2-x-2=(x-2)(x+1)$ has roots $2,-1$, and $x^2-2x-8=(x-4)(x+2)$ has roots $4,-2$ — exactly twice as large.

Watch out. Doubling both roots doubles their sum but *quadruples* their product. Writing $n=2m$ instead of $n=4m$ gives $\frac{n}{p}=\frac{2m}{m/2}=4$, choice (C).

Another way. If $f(x)=x^2+px+m$ has roots r,s, then $f(x/2)$ vanishes at $x=2r,2s$. Multiplying by 4 to make it monic, the equation with the doubled roots is

$$4f\left(\tfrac{x}{2}\right)=x^2+2px+4m.$$

Matching with x^2+mx+n gives $m=2p$ and $n=4m=8p$ at once.

Answer: (D) 8

4 Counting & Probability

Source: 2024 AMC 10B #16

> Jerry likes to play with numbers. One day, he wrote all the integers from 1 to 2024 on the whiteboard. Then he repeatedly chose four numbers on the whiteboard, erased them, and replaced them by either their sum or their product. (For example, Jerry's first step might have been to erase 1, 2, 3, and 5, and then write either 11, their sum, or 30, their product, on the whiteboard.) After repeatedly performing this operation, Jerry noticed that all the remaining numbers on the whiteboard were odd. What is the maximum possible number of integers on the whiteboard at that time?
>
> **(A)** 1010 **(B)** 1011 **(C)** 1012 **(D)** 1013 **(E)** 1014

Key idea. Every move removes exactly 3 numbers from the board (four out, one in), so keeping many numbers means making *few* moves. The question is about odd and even, so count how many even numbers one move can get rid of.

Step 1: the board shrinks by 3 per move. After m moves there are $2024 - 3m$ numbers. We want m as small as possible.

Step 2: one move removes at most 3 evens. At the start there are 1012 even and 1012 odd numbers. Suppose a move erases e even numbers. If it writes an odd number, the even count drops by e; if it writes an even number, the even count drops by $e - 1$. Four evens always give an even sum and an even product, so $e = 4$ drops the count by only 3. In every case the drop is at most 3.

Step 3: a lower bound on moves. All 1012 evens must disappear, so $3m \geq 1012$, giving $m \geq 338$ (since $3 \cdot 337 = 1011$). Hence at most $2024 - 3 \cdot 338 = 1010$ numbers remain.

Step 4: 1010 is achievable. Since $1012 = 3 \cdot 337 + 1$, make 337 moves that each take three evens and one odd and write their *sum*, which is odd. Now one even number is left; take it with three odd numbers and write the sum, again odd. That is 338 moves, and all 1010 remaining numbers are odd.

Watch out. Answering 1012, choice (C), pictures the even numbers simply vanishing while all 1012 odd numbers survive. But no move can delete an even number on its own: each move erases four numbers and writes one, so the board loses 3 entries every time, and at least 338 moves are needed.

Answer: (A) 1010

5 Geometry

Source: 2007 AMC 10B #17

Point P is inside equilateral $\triangle ABC$. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB?

(A) 4 **(B)** $3\sqrt{3}$ **(C)** 6 **(D)** $4\sqrt{3}$ **(E)** 9

Key idea. Three perpendiculars from one inside point to the three sides suggest splitting $\triangle ABC$ into three triangles with apex P: each perpendicular is a height, and the three areas must add up to the whole.

Step 1. Let s be the side length. Joining P to A, B, C cuts the triangle into $\triangle APB$, $\triangle BPC$, $\triangle CPA$, each with base s and heights 1, 2, 3. Their areas add to

$$\tfrac{1}{2}s \cdot 1 + \tfrac{1}{2}s \cdot 2 + \tfrac{1}{2}s \cdot 3 = \tfrac{1}{2}s(1+2+3) = 3s.$$

Step 2. The area of an equilateral triangle with side s is $\frac{\sqrt{3}}{4}s^2$. So

$$\frac{\sqrt{3}}{4}s^2 = 3s \quad \Longrightarrow \quad s = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$$

This also shows a nice fact: the three distances always add up to the height of the triangle. Here the height is $1+2+3=6$, and an equilateral triangle of height 6 has side $6 \cdot \frac{2}{\sqrt{3}} = 4\sqrt{3}$.

Watch out. The sum $1+2+3=6$ is the *height*, not the side. Stopping there gives choice (C).

Answer: (D) $4\sqrt{3}$

6 Algebra

Source: 2025 AMC 10A #18

The harmonic mean of a collection of numbers is the reciprocal of the arithmetic mean of the reciprocals of the numbers in the collection. For example, the harmonic mean of $4, 4$, and 5 is

$$\frac{1}{\frac{1}{3}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{5}\right)} = \frac{30}{7}.$$

What is the harmonic mean of all the real roots of the 4050th degree polynomial

$$\prod_{k=1}^{2025}\left(kx^2-4x-3\right) = \left(x^2-4x-3\right)\cdot\left(2x^2-4x-3\right) \cdot\left(3x^2-4x-3\right)\cdots \cdot\left(2025x^2-4x-3\right)?$$

(A) $-\frac{5}{3}$ **(B)** $-\frac{3}{2}$ **(C)** $-\frac{6}{5}$ **(D)** $-\frac{5}{6}$ **(E)** $-\frac{2}{3}$

Key idea. The harmonic mean needs only the *sum of the reciprocals* of the roots and the *number* of roots. For a quadratic, $\frac{1}{x_1}+\frac{1}{x_2}=\frac{x_1+x_2}{x_1x_2}$, which Vieta's formulas hand you without solving anything.

Step 1: count the roots. The factor kx^2-4x-3 has discriminant $16+12k>0$, so it has two distinct real roots. A root shared by factors $j\neq k$ would satisfy $jx^2-4x-3=kx^2-4x-3$, so $(j-k)x^2=0$ and $x=0$; but $x=0$ gives $-3\neq 0$. So all $2\cdot 2025=4050$ real roots are distinct.

Step 2: reciprocals from one factor. For kx^2-4x-3, Vieta gives $x_1+x_2=\frac{4}{k}$ and $x_1x_2=-\frac{3}{k}$. So

$$\frac{1}{x_1}+\frac{1}{x_2}=\frac{x_1+x_2}{x_1x_2}=\frac{4/k}{-3/k}=-\frac{4}{3},$$

the same for every k.

Step 3: add up. The sum of all 4050 reciprocals is $2025\cdot\left(-\frac{4}{3}\right)=-2700$. Their arithmetic mean is $\frac{-2700}{4050}=-\frac{2}{3}$, and the harmonic mean is its reciprocal:

$$\frac{4050}{-2700}=-\frac{3}{2}.$$

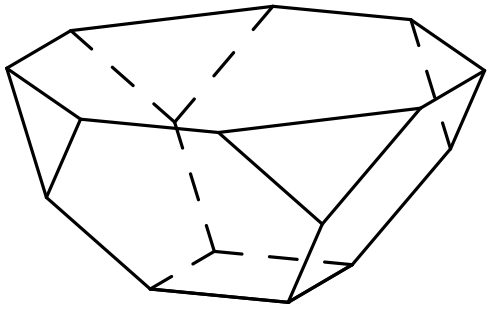
Watch out. Stopping at the mean of the reciprocals, $-\frac{2}{3}$, which is choice (E). The definition asks for the *reciprocal* of that mean.

Answer: (B) $-\frac{3}{2}$

7 Geometry

Source: 2022 AMC 10A #21

A bowl is formed by attaching four regular hexagons of side 1 to a square of side 1. The edges of the adjacent hexagons coincide, as shown in the figure. What is the area of the octagon obtained by joining the top eight vertices of the four hexagons, situated on the rim of the bowl?



(A) 6 **(B)** 7 **(C)** $5+2\sqrt{2}$ **(D)** 8 **(E)** 9

Key idea. By symmetry all eight rim vertices are at the same height, so the octagon is horizontal and its area equals the area of its shadow seen from directly above. The only unknown is how steeply the hexagons tilt — and the condition that neighboring hexagons share an edge pins that down.

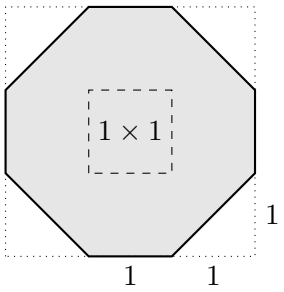
Step 1: coordinates. Put the square on the floor with corners $\left(\pm\frac{1}{2},\pm\frac{1}{2},0\right)$. Look at the front hexagon, attached along the side $y=-\frac{1}{2}$. Inside its own plane, let $\vec{w}$ be the unit vector perpendicular to the attached side, pointing away from the square, and write $\vec{w}=(0,-a,b)$ with $a^2+b^2=1$; here a is its horizontal (outward) part.

In a regular hexagon of side 1, the edge leaving the corner $P=\left(\frac{1}{2},-\frac{1}{2},0\right)$ goes $\frac{1}{2}$ along the attached side and $\frac{\sqrt{3}}{2}$ along $\vec{w}$. So it ends at

$$P+\left(\tfrac{1}{2},\,-\tfrac{\sqrt{3}}{2}a,\,\tfrac{\sqrt{3}}{2}b\right).$$

Step 2: the shared edge fixes the tilt. The right hexagon is the same picture turned 90°; its edge from P ends at $P+\left(\frac{\sqrt{3}}{2}a,\,-\frac{1}{2},\,\frac{\sqrt{3}}{2}b\right)$. These edges coincide, so $\frac{\sqrt{3}}{2}a=\frac{1}{2}$, giving $a=\frac{1}{\sqrt{3}}$ and $b=\sqrt{1-\frac{1}{3}}=\sqrt{\frac{2}{3}}$.

Step 3: the rim. The top side of the front hexagon is $\sqrt{3}$ away from the attached side along $\vec{w}$, so it sits $\sqrt{3}\cdot a=1$ unit further out horizontally than the square's side, at height $\sqrt{3}\,b=\sqrt{2}$. Its endpoints project to $\left(\pm\frac{1}{2},-\frac{3}{2}\right)$. By symmetry the eight rim vertices are $\left(\pm\frac{1}{2},\pm\frac{3}{2}\right)$ and $\left(\pm\frac{3}{2},\pm\frac{1}{2}\right)$, all at height $\sqrt{2}$.



Step 4: area. Seen from above, the octagon is a 3×3 square with four isosceles right triangles of leg 1 cut from the corners:

$$3^2 - 4 \cdot \tfrac{1}{2} \cdot 1^2 = 9 - 2 = 7.$$

Since the octagon is horizontal, this is its true area.

Watch out. The octagon is not regular: its sides alternate between 1 (hexagon tops) and $\sqrt{2}$ (the gaps between neighboring hexagons). Treating it as a regular octagon of side 1 gives $2+2\sqrt{2} \approx 4.83$, which matches no choice.

Answer: (B) 7

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8 Counting & Probability

Source: 2011 AMC 10A #21

Two counterfeit coins of equal weight are mixed with 8 identical genuine coins. The weight of each of the counterfeit coins is different from the weight of each of the genuine coins. A pair of coins is selected at random without replacement from the 10 coins. A second pair is selected at random without replacement from the remaining 8 coins. The combined weight of the first pair is equal to the combined weight of the second pair. What is the probability that all 4 selected coins are genuine?

(A) $\frac{7}{11}$ **(B)** $\frac{9}{13}$ **(C)** $\frac{11}{15}$ **(D)** $\frac{15}{19}$ **(E)** $\frac{15}{16}$

Key idea. This is a conditional probability: we are *told* the two pairs weigh the same. So list every way the weights can match, count the outcomes in each case, and take the fraction belonging to the "all genuine" case.

Step 1: when can the weights match? Let a genuine coin weigh g and a counterfeit weigh $c \neq g$. A pair weighs $2g$, $g + c$, or $2c$, and these three totals are all different. The two counterfeits cannot both be in one pair, since the other pair would then weigh $2g \neq 2c$. Nor can exactly one counterfeit be among the four coins: its pair would weigh $g + c$ and the other $2g$. So the weights match in exactly two ways:

- both pairs are genuine ($2g$ and $2g$), or
- each pair holds one counterfeit and one genuine coin ($g + c$ and $g + c$).

Step 2: count ordered outcomes. Every choice of (first pair, second pair) is equally likely, so we count them.

All genuine: the first pair is 2 of the 8 genuine coins, $\binom{8}{2} = 28$ ways; the second is 2 of the remaining 6, $\binom{6}{2} = 15$ ways. Total $28 \cdot 15 = 420$.

One counterfeit each: the first pair takes one of 2 counterfeits and one of 8 genuine coins, $2 \cdot 8 = 16$ ways; the second pair must take the other counterfeit and one of the 7 remaining genuine coins, 7 ways. Total $16 \cdot 7 = 112$.

Step 3: the probability. Given that the weights match,

$$P = \frac{420}{420 + 112} = \frac{420}{532} = \frac{15}{19}.$$

(Counting unordered sets of two pairs instead halves both counts, 210 and 56, and gives the same fraction.)

Watch out. In the mixed case it is easy to forget that the counterfeit coins can go with their genuine partners in two ways; counting it as 56 next to 420 (or 28 next to 210) gives $\frac{15}{17}$, which is not a choice. Whatever you do, count both cases the same way — ordered or unordered.

Answer: (D) $\frac{15}{19}$

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9 Number Theory

Source: 2018 AMC 10B #23

How many ordered pairs (a, b) of positive integers satisfy the equation

$$a \cdot b + 63 = 20 \cdot \text{lcm}(a, b) + 12 \cdot \gcd(a, b),$$

where $\gcd(a, b)$ denotes the greatest common divisor of a and b, and $\text{lcm}(a, b)$ denotes their least common multiple?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Key idea. An equation mixing ab, $\text{lcm}(a, b)$ and $\gcd(a, b)$ calls for the identity $ab = \gcd(a, b) \cdot \text{lcm}(a, b)$: it turns everything into two unknowns. The result is "product = linear terms + constant," which factors by Simon's Favorite Factoring Trick.

Step 1: two new unknowns. Let $g = \gcd(a, b)$ and $\ell = \text{lcm}(a, b)$. Since $ab = g\ell$, the equation becomes

$$g\ell + 63 = 20\ell + 12g.$$

Step 2: factor. Move terms to get $g\ell - 20\ell - 12g = -63$, and add $20 \cdot 12 = 240$ to both sides:

$$g\ell - 20\ell - 12g + 240 = 177 \quad \Longrightarrow \quad (\ell - 12)(g - 20) = 177 = 3 \cdot 59.$$

Step 3: list the possibilities. Both g and ℓ are positive, so $g - 20 \geq -19$ and $\ell - 12 \geq -11$. If both factors were negative, one of them would have to be -59 or -177, which is impossible. So both factors are positive:

$$(\ell - 12,\ g - 20) \in \{(1, 177), (3, 59), (59, 3), (177, 1)\},$$

giving $(\ell, g) = (13, 197)$, $(15, 79)$, $(71, 23)$, $(189, 21)$.

Step 4: the gcd must divide the lcm. In the first two, $g > \ell$, which is impossible. In the third, 23 does not divide 71. Only $(\ell, g) = (189, 21)$ survives, and indeed $189 = 9 \cdot 21$.

Step 5: recover a and b. Write $a = 21u$ and $b = 21v$ with $\gcd(u, v) = 1$. Then $\operatorname{lcm}(a, b) = 21uv = 189$, so $uv = 9$. The coprime ordered pairs with product 9 are $(1, 9)$ and $(9, 1)$, giving $(a, b) = (21, 189)$ and $(189, 21)$.

Check: $21 \cdot 189 + 63 = 3969 + 63 = 4032$ and $20 \cdot 189 + 12 \cdot 21 = 3780 + 252 = 4032$.

Watch out. The pair $(u, v) = (3, 3)$ also has product 9, but it is not coprime: $a = b = 63$ has gcd = lcm = 63, not 21 and 189. Including it would give 3 pairs, which is not a choice.

Answer: (B) 2

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10 Algebra

Source: 2023 AMC 10B #23

An arithmetic sequence of positive integers has $n > 3$ terms, initial term a, and common difference $d > 1$. Carl wrote down all the terms in this sequence correctly except for one term, which was off by 1. The sum of the terms he wrote down was 222. What is $a + d + n$?

(A) 24 **(B)** 20 **(C)** 22 **(D)** 28 **(E)** 26

Key idea. One term off by 1 means the true sum is 221 or 223. The sum formula $2S = n\big(2a+(n-1)d\big)$ says n divides $2S$, and the conditions $a \geq 1$, $d \geq 2$ keep n small — so only a few divisors need checking.

Step 1: the true sum. Changing one term by 1 changes the sum by 1, so the true sum S satisfies $S = 222 \pm 1$, i.e. $S = 221$ or $S = 223$.

Step 2: the key equation. The terms are $a, a + d, \ldots, a + (n - 1)d$, so

$$2S = n\big(2a + (n - 1)d\big).$$

The bracket is a whole number, so n divides $2S$, which is 442 or 446.

Step 3: n is small. Since $a \geq 1$ and $d \geq 2$, the bracket is at least $2 + 2(n - 1) = 2n$. So $2S \geq 2n^2$, i.e. $n^2 \leq S \leq 223$, giving $n \leq 14$. Together with $n > 3$: $4 \leq n \leq 14$.

Step 4: test the divisors.

- $446 = 2 \cdot 223$ with 223 prime: its divisors are $1, 2, 223, 446$, none between 4 and 14.
- $442 = 2 \cdot 13 \cdot 17$: its divisors are $1, 2, 13, 17, 26, 34, 221, 442$; the only one between 4 and 14 is 13.

So $S = 221$ and $n = 13$.

Step 5: find a and d. Now $442 = 13(2a + 12d)$, so $2a + 12d = 34$, i.e. $a + 6d = 17$. With $d \geq 2$: $d = 2$ gives $a = 5$, while $d \geq 3$ would make $a \leq -1$. So $a = 5$, $d = 2$, and the sequence $5, 7, \ldots, 29$ has sum $13 \cdot \frac{5+29}{2} = 221$. ✓

Therefore $a + d + n = 5 + 2 + 13 = 20$.

Watch out. Forgetting $d > 1$. Then $d = 1$, $a = 11$ also solves $a + 6d = 17$, giving $a + d + n = 25$ — not a choice, which is a hint that a condition was dropped.

Answer: (B) 20