AMC 10

SET 27

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 13–23 of past AMC 10 contests (2000–2025) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**D**	2017 AMC 10A #13	Algebra
2	**C**	2023 AMC 10B #14	Number Theory
3	**D**	2024 AMC 10A #16	Geometry
4	**C**	2015 AMC 10B #16	Counting & Probability
5	**A**	2006 AMC 10A #17	Geometry
6	**B**	2020 AMC 10B #18	Counting & Probability
7	**E**	2018 AMC 10A #21	Geometry
8	**C**	2013 AMC 10B #21	Algebra
9	**C**	2010 AMC 10B #22	Counting & Probability
10	**A**	2016 AMC 10A #23	Algebra

1 Algebra

Source: 2017 AMC 10A #13

Define a sequence recursively by $F_0 = 0,\ F_1 = 1$, and $F_n =$ the remainder when $F_{n-1} + F_{n-2}$ is divided by 3, for all $n \geq 2$. Thus the sequence starts $0, 1, 1, 2, 0, 2, \ldots$. What is

$$F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}?$$

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Key idea. A sequence built from the previous two terms, using only the values $0, 1, 2$, must eventually repeat a pair of consecutive terms. Once a pair repeats, the whole sequence repeats. Indices like 2017 are a strong hint to look for that cycle.

Step 1. Keep going with remainders mod 3:

$$F_0, \ldots, F_9 = 0,\ 1,\ 1,\ 2,\ 0,\ 2,\ 2,\ 1,\ 0,\ 1.$$

For example, F_6 is the remainder of $F_5 + F_4 = 2 + 0 = 2$, so $F_6 = 2$; and F_7 is the remainder of $2 + 2 = 4$, so $F_7 = 1$. Likewise F_8 is the remainder of $1 + 2 = 3$, which is 0, and F_9 is the remainder of $0 + 1$, which is 1.

Step 2. The pair $(F_8, F_9) = (0, 1)$ equals the starting pair (F_0, F_1). Each term depends only on the two before it, so from here on the sequence copies itself: it repeats every 8 terms.

Step 3. The sum asked for is of 8 consecutive terms, F_{2017} through F_{2024}. Any 8 consecutive terms of an 8-periodic sequence are one full cycle in some order, so the sum equals one full cycle:

$$0 + 1 + 1 + 2 + 0 + 2 + 2 + 1 = 9.$$

To see it directly: $2017 = 8 \cdot 252 + 1$, so $F_{2017}, \ldots, F_{2024}$ equal $F_1, \ldots, F_8 = 1, 1, 2, 0, 2, 2, 1, 0$, whose sum is also 9.

Answer: (D) 9

2 Number Theory

Source: 2023 AMC 10B #14

How many ordered pairs of integers (m, n) satisfy the equation

$$m^2 + mn + n^2 = m^2n^2?$$

(A) 7 **(B)** 1 **(C)** 3 **(D)** 6 **(E)** 5

Key idea. The right side m^2n^2 grows much faster than the left side $m^2 + mn + n^2$ once both numbers are big, so only small values can work. Bound the smaller of $|m|, |n|$ and then check the few candidates.

Step 1: a zero. If $m = 0$ the equation becomes $n^2 = 0$, so $n = 0$. Likewise $n = 0$ forces $m = 0$. This gives the pair $(0, 0)$.

Step 2: both nonzero. Suppose $|m| \le |n|$ (the other case is the same with the roles swapped). Then $m^2 \le n^2$ and $mn \le |m||n| \le n^2$, so

$$m^2n^2 = m^2 + mn + n^2 \le n^2 + n^2 + n^2 = 3n^2.$$

Dividing by $n^2 > 0$ gives $m^2 \le 3$, so $m = 1$ or $m = -1$.

Step 3: the candidates. If $m = 1$: $1 + n + n^2 = n^2$, so $n = -1$. If $m = -1$: $1 - n + n^2 = n^2$, so $n = 1$. Swapping roles produces the same two pairs $(1, -1)$ and $(-1, 1)$; both check, e.g. $1 - 1 + 1 = 1 = 1 \cdot 1$.

So the solutions are $(0, 0)$, $(1, -1)$, $(-1, 1)$: three pairs.

Another way. Add mn to both sides: $(m + n)^2 = m^2n^2 + mn = mn(mn + 1)$. For an integer $k = mn$, $k(k + 1)$ is a perfect square only when $k = 0$ or $k = -1$: if $k \ge 1$ then $k^2 < k(k + 1) < (k + 1)^2$, and if $k \le -2$ then $k(k + 1) = |k|(|k| - 1)$ lies strictly between $(|k| - 1)^2$ and $|k|^2$. So $mn = 0$, giving $(0, 0)$, or $mn = -1$, giving $(1, -1)$ and $(-1, 1)$.

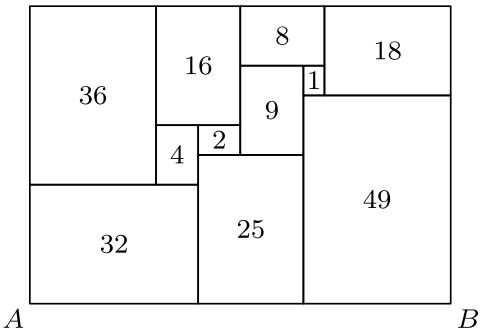
Watch out. The pair $(0, 0)$ is easy to forget; without it you count 2, which is not a choice.

Answer: (C) 3

3 Geometry

Source: 2024 AMC 10A #16

All of the rectangles in the figure below, which is drawn to scale, are similar to the enclosing rectangle. Each number represents the area of the rectangle. What is length AB?



(A) $4+4\sqrt{5}$ **(B)** $10\sqrt{2}$ **(C)** $5+5\sqrt{5}$ **(D)** $10\sqrt[4]{8}$ **(E)** 20

Key idea. Similar rectangles have sides in the ratio of the *square roots* of their areas. The total area is easy to find, so every small side can be written as a multiple of the big rectangle's sides. Then one edge of the big rectangle, made of two pieces, gives an equation for its shape.

Step 1. Add the areas: $36+16+8+18+1+9+2+4+25+49+32=200$. Let the enclosing rectangle have width $w=AB$ (the long side) and height h, so $wh=200$.

Step 2. A rectangle of area S similar to the big one is a copy scaled by $\sqrt{S/200}$. Its long side is $w\sqrt{S/200}$ and its short side is $h\sqrt{S/200}$. From the figure, the 36 rectangle stands upright (long side vertical), while the 32 rectangle lies flat (short side vertical).

Step 3. The left edge consists of the 36 and the 32 rectangles stacked:

$$h = w\sqrt{\tfrac{36}{200}} + h\sqrt{\tfrac{32}{200}} = \tfrac{3\sqrt{2}}{10}\,w + \tfrac{2}{5}\,h.$$

So $\frac{3}{5}h=\frac{3\sqrt{2}}{10}w$, which gives $h=\frac{\sqrt{2}}{2}w=\frac{w}{\sqrt{2}}$.

Step 4. Substitute into $wh=200$: $\dfrac{w^2}{\sqrt{2}}=200$, so $w^2=200\sqrt{2}$. Then

$$w=\sqrt{100\cdot 2\sqrt{2}}=10\sqrt{2^{3/2}}=10\cdot 2^{3/4}=10\sqrt[4]{8}.$$

Another way. The bottom edge works too. The 32 rectangle lies flat, while the 25 and 49 rectangles stand upright, so $w=\frac{2}{5}w+\frac{5}{10\sqrt{2}}h+\frac{7}{10\sqrt{2}}h$. This gives $\frac{3}{5}w=\frac{6}{5\sqrt{2}}h$, i.e. $w=\sqrt{2}\,h$ — the same shape.

Watch out. Treating the enclosing rectangle as a square gives $AB=\sqrt{200}=10\sqrt{2}$, choice (B). The figure is clearly wider than it is tall: in fact $w=\sqrt{2}\,h$.

Answer: (D) $10\sqrt[4]{8}$

4 Counting & Probability

Source: 2015 AMC 10B #16

Al, Bill, and Cal will each randomly be assigned a whole number from 1 to 10, inclusive, with no two of them getting the same number. What is the probability that Al's number will be a whole number multiple of Bill's and Bill's number will be a whole number multiple of Cal's?

(A) $\frac{9}{1000}$ **(B)** $\frac{1}{90}$ **(C)** $\frac{1}{80}$ **(D)** $\frac{1}{72}$ **(E)** $\frac{2}{121}$

Key idea. There are only $10 \cdot 9 \cdot 8$ equally likely assignments, and the favorable ones are a short chain "Cal divides Bill divides Al" among small numbers — so list the chains, organized by Cal's number, the smallest link.

Step 1: count all assignments. Al has 10 choices, then Bill 9 (not Al's number), then Cal 8. So there are $10 \cdot 9 \cdot 8 = 720$ equally likely assignments.

Step 2: list the chains. Write (A, B, C) for the numbers of Al, Bill, Cal. Since the numbers are different, B is a multiple of C that is at least $2C$, and A is a multiple of B that is at least $2B$. So $A \geq 4C$, which forces $C \leq 2$.

$C = 1$. Then $B \geq 2$ and $A \geq 2B$, so $B \leq 5$: $B = 2$: $A \in \{4, 6, 8, 10\}$; $B = 3$: $A \in \{6, 9\}$; $B = 4$: $A = 8$; $B = 5$: $A = 10$. That is $4 + 2 + 1 + 1 = 8$ chains.

$C = 2$. Then $B \geq 4$ is even and $A \geq 2B$ gives $B \leq 5$, so $B = 4$ and $A = 8$: one chain, $(8, 4, 2)$.

Step 3. There are $8 + 1 = 9$ favorable assignments, so the probability is

$$\frac{9}{720} = \frac{1}{80}.$$

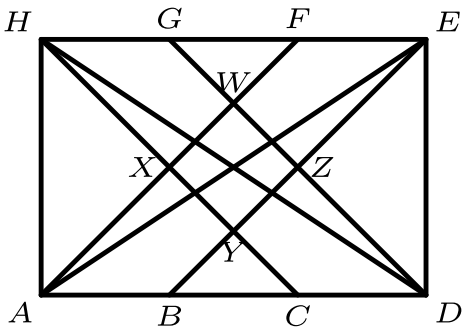
Watch out. Two slips land on wrong choices. Forgetting the chain $(8, 4, 2)$ gives $\frac{8}{720} = \frac{1}{90}$, choice (B). Using $10^3 = 1000$ total assignments — as if repeats were allowed — gives $\frac{9}{1000}$, choice (A); but the problem says no two get the same number.

Answer: (C) $\frac{1}{80}$

5 Geometry

Source: 2006 AMC 10A #17

In rectangle $ADEH$, points B and C trisect $\overline{AD}$, and points G and F trisect $\overline{HE}$. In addition, $AH = AC = 2$. What is the area of quadrilateral $WXYZ$ shown in the figure?



(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{\sqrt{3}}{2}$ **(D)** $\frac{2\sqrt{2}}{3}$ **(E)** $\frac{2\sqrt{3}}{3}$

Key idea. All the points sit on a neat grid, so put the rectangle on coordinate axes. Each vertex of $WXYZ$ is then the crossing of two lines with simple equations. The figure is crowded, but only four of its segments actually bound $WXYZ$.

Step 1: coordinates. Since B and C trisect $\overline{AD}$ and $AC = 2$, each third is 1 and $AD = 3$. Put $A = (0,0)$, $D = (3,0)$, $E = (3,2)$, $H = (0,2)$. Then $B = (1,0)$, $C = (2,0)$, $G = (1,2)$, $F = (2,2)$.

Step 2: the four lines. From the figure, $WXYZ$ is bounded by segments $\overline{AF}$, $\overline{HC}$, $\overline{BE}$, $\overline{GD}$:

$$AF:\ y = x, \quad HC:\ y = 2 - x, \quad BE:\ y = x - 1, \quad GD:\ y = 3 - x.$$

Step 3: the vertices. W: $x = 3 - x$, so $W = (1.5, 1.5)$. X: $x = 2 - x$, so $X = (1, 1)$. Y: $2 - x = x - 1$, so $Y = (1.5, 0.5)$. Z: $x - 1 = 3 - x$, so $Z = (2, 1)$.

Step 4: the area. The sides have slopes ± 1, so neighbouring sides are perpendicular, and each side has length $\frac{\sqrt{2}}{2}$; $WXYZ$ is a square. Its diagonals WY and XZ both have length 1, so its area is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$.

Watch out. The diagonal of the square is 1, not its side; squaring the diagonal gives area 1, which is not a choice.

Answer: (A) $\frac{1}{2}$

6 Counting & Probability

Source: 2020 AMC 10B #18

> An urn contains one red ball and one blue ball. A box of extra red and blue balls lies nearby. George performs the following operation four times: he draws a ball from the urn at random and then takes a ball of the same color from the box and returns those two matching balls to the urn. After the four iterations the urn contains six balls. What is the probability that the urn contains three balls of each color?
>
> **(A)** $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

Key idea. The urn starts with one of each color and gains one ball per draw, so ending at 3 and 3 means exactly two red draws and two blue draws. Compute the probability of one such order, then notice every order gives the same product.

Step 1. Before the four draws the urn holds 2, 3, 4, 5 balls, so every order has denominator $2\cdot3\cdot4\cdot5 = 120$.

Step 2. Take the order red, red, blue, blue. The first red draw has 1 red ball available, the second has 2; then the first blue draw has 1 blue ball, the second 2:

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}\cdot\frac{2}{5} = \frac{1\cdot2\cdot1\cdot2}{120} = \frac{4}{120} = \frac{1}{30}.$$

Step 3. In *any* order with two reds and two blues, the red numerators are still 1 then 2 and the blue numerators are still 1 then 2; only their positions move. So each order has probability $\frac{1}{30}$.

Step 4. There are $\binom{4}{2} = 6$ orders, so the probability is $6\cdot\frac{1}{30} = \frac{1}{5}$.

Watch out. Treating every draw as a 50–50 coin flip gives $\binom{4}{2}\cdot\frac{1}{16} = \frac{3}{8}$, which is not a choice. The draws are not equally likely: whichever color is drawn becomes more likely next time.

Another way. The same computation shows that ending with k red draws out of 4 has probability $\binom{4}{k}\cdot\frac{k!\,(4-k)!}{120} = \frac{24}{120} = \frac{1}{5}$ for every $k = 0, 1, 2, 3, 4$. All five outcomes are equally likely, so the answer is $\frac{1}{5}$.

Answer: (B) $\frac{1}{5}$

7 Geometry

Source: 2018 AMC 10A #21

Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy-plane intersect at exactly 3 points?

(A) $a = \frac{1}{4}$ **(B)** $\frac{1}{4} < a < \frac{1}{2}$ **(C)** $a > \frac{1}{4}$ **(D)** $a = \frac{1}{2}$ **(E)** $a > \frac{1}{2}$

Key idea. Intersection points are solutions of the system, so substitute the parabola into the circle. The parabola gives y as a function of x, so each real x-solution gives exactly one point, and we just count distinct real x-values. The equation that comes out has only even powers of x, which makes the count easy.

Step 1: substitute. A point is on both curves exactly when $y = x^2 - a$ and $x^2 + (x^2 - a)^2 = a^2$. Expand the second equation:

$$x^2 + x^4 - 2ax^2 + a^2 = a^2 \quad \Longrightarrow \quad x^4 - (2a-1)x^2 = 0 \quad \Longrightarrow \quad x^2\big(x^2 - (2a-1)\big) = 0.$$

Step 2: count the x-values. The factor $x^2 = 0$ always gives $x = 0$, the point $(0, -a)$. (It is on the circle: $0^2 + (-a)^2 = a^2$.) The second factor gives $x^2 = 2a - 1$.

- If $2a - 1 > 0$, that is $a > \frac{1}{2}$, then $x = \pm\sqrt{2a-1}$ are two more distinct nonzero values. Total: 3 points.
- If $2a - 1 = 0$, then $x^2 = 0$ again: no new point. Total: 1 point.
- If $2a - 1 < 0$, then $x^2 = 2a - 1$ has no real solution. Total: 1 point.

Since y is determined by x, distinct x-values give distinct points, and no two points share an x-value.

Step 3. Exactly 3 intersection points occur precisely when $a > \frac{1}{2}$.

For example, at $a = 1$ the points are $(0, -1)$ and $(\pm 1, 0)$: each satisfies $x^2 + y^2 = 1$ and $y = x^2 - 1$.

Watch out. At the boundary value $a = \frac{1}{2}$ the two extra points merge into $(0, -\frac{1}{2})$: then $x^2 + y^2 - a^2 = x^4 > 0$ for every $x \neq 0$, so the parabola only touches the circle at its vertex and otherwise stays outside it, giving just 1 intersection point. Choosing (D) mistakes this tangency for the three-point case.

Answer: (E) $a > \frac{1}{2}$

8 Algebra

Source: 2013 AMC 10B #21

> Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. What is the smallest possible value of N?
>
> **(A)** 55 **(B)** 89 **(C)** 104 **(D)** 144 **(E)** 273

Key idea. Everything in a Fibonacci-type sequence is fixed by its first two terms, so write the seventh term as a formula in them. Two sequences with the same seventh term then give one equation in whole numbers, and divisibility does the rest.

Step 1: the seventh term. Starting from u, v, the terms are

$$u,\ v,\ u+v,\ u+2v,\ 2u+3v,\ 3u+5v,\ 5u+8v.$$

So $N = 5u + 8v$, and "non-decreasing" means $u \le v$.

Step 2: compare the two sequences. Let them start a_1, a_2 and b_1, b_2 with $a_1 < b_1$. From $5a_1 + 8a_2 = 5b_1 + 8b_2$,

$$5(b_1 - a_1) = 8(a_2 - b_2).$$

The left side is a multiple of 8, and 5 shares no factor with 8, so 8 divides $b_1 - a_1$. Since $b_1 - a_1 > 0$, we get $b_1 - a_1 \ge 8$, so $b_1 \ge 8$.

Step 3: the lower bound. The second sequence is non-decreasing, so $b_2 \ge b_1 \ge 8$. Hence

$$N = 5b_1 + 8b_2 \ge 5 \cdot 8 + 8 \cdot 8 = 104.$$

Step 4: it is achieved. Take $b_1 = b_2 = 8$ and $a_1 = 0$, $a_2 = 13$:

$$0,\ 13,\ 13,\ 26,\ 39,\ 65,\ 104 \qquad \text{and} \qquad 8,\ 8,\ 16,\ 24,\ 40,\ 64,\ 104.$$

Both are non-decreasing, their first terms differ, and both seventh terms are 104. So the smallest N is 104.

Watch out. Ignoring "non-decreasing". Then $(8, 0)$ and $(0, 5)$ would both give $N = 40$, which is not even a choice. The condition $b_2 \ge b_1$ is what pushes the answer up to 104.

Answer: (C) 104

9 Counting & Probability

Source: 2010 AMC 10B #22

Seven distinct pieces of candy are to be distributed among three bags. The red bag and the blue bag must each receive at least one piece of candy; the white bag may remain empty. How many arrangements are possible?

(A) 1930 **(B)** 1931 **(C)** 1932 **(D)** 1933 **(E)** 1934

Key idea. "At least one" conditions are easiest to handle by counting what is *forbidden* and subtracting. Two conditions can fail at the same time, so use inclusion–exclusion.

Step 1. All distributions. Each of the 7 distinct candies independently goes into one of 3 bags, so there are $3^7 = 2187$ distributions in all.

Step 2. Red bag empty. Then each candy has only 2 choices (blue or white): $2^7 = 128$ distributions. In the same way, 128 distributions leave the blue bag empty.

Step 3. Both empty. Then every candy goes into the white bag: 1 distribution. It was counted in both cases of Step 2, so the number of bad distributions is

$$128 + 128 - 1 = 255.$$

Step 4. The number of allowed arrangements is

$$2187 - 255 = 1932.$$

Watch out. Subtracting $128 + 128$ without adding back the one distribution with both bags empty removes that distribution twice and gives $2187 - 256 = 1931$, choice (B).

Answer: (C) 1932

10 Algebra

Source: 2016 AMC 10A #23

A binary operation $\diamondsuit$ has the properties that

$$a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$$

and that $a \diamondsuit a = 1$ for all nonzero real numbers a, b, and c. (Here $\cdot$ represents multiplication). The solution to the equation

$$2016 \diamondsuit (6 \diamondsuit x) = 100$$

can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

(A) 109 **(B)** 201 **(C)** 301 **(D)** 3049 **(E)** 33,601

Key idea. The rule $a \diamondsuit a = 1$ is begging to be used inside the first identity: choose the variables so that some $\diamondsuit$ collapses to 1. Doing this twice pins down a formula for $a \diamondsuit b$.

Step 1. Put $c = b$ in $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$. Since $b \diamondsuit b = 1$,

$$a \diamondsuit 1 = (a \diamondsuit b) \cdot b. \tag{$*$}$$

Step 2. Now put $b = a$ in $(*)$: $a \diamondsuit 1 = (a \diamondsuit a) \cdot a = 1 \cdot a = a$. Putting this back into $(*)$ gives $a = (a \diamondsuit b) \cdot b$, so

$$a \diamondsuit b = \frac{a}{b}.$$

Check: $a \diamondsuit (b \diamondsuit c) = \dfrac{a}{b/c} = \dfrac{ac}{b} = \dfrac{a}{b} \cdot c$, and $\frac{a}{a} = 1$. ✓

Step 3. Then $6 \diamondsuit x = \frac{6}{x}$ and

$$2016 \diamondsuit \frac{6}{x} = \frac{2016}{6/x} = \frac{2016x}{6} = 336x.$$

Setting $336x = 100$ gives $x = \frac{100}{336} = \frac{25}{84}$ (divide top and bottom by 4; $25 = 5^2$ and $84 = 2^2 \cdot 3 \cdot 7$ share no factor).

So $p + q = 25 + 84 = 109$.

Watch out. Stopping at $\frac{100}{336}$ without reducing gives 436, which is not a choice; the problem insists p and q be relatively prime.

Answer: (A) 109