AMC 10

HALF-LENGTH PRACTICE

SET 28

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 13–23 of past AMC 10 contests (2000–2025) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**B**	2024 AMC 10B #13	Algebra
2	**D**	2020 AMC 10B #14	Geometry
3	**D**	2021 Fall AMC 10B #16	Counting & Probability
4	**A**	2012 AMC 10B #16	Geometry
5	**E**	2020 AMC 10A #17	Algebra
6	**E**	2023 AMC 10A #19	Geometry
7	**D**	2021 AMC 10A #20	Counting & Probability
8	**C**	2018 AMC 10B #21	Number Theory
9	**D**	2024 AMC 10A #23	Algebra
10	**C**	2017 AMC 10B #23	Number Theory

1 Algebra

Source: 2024 AMC 10B #13

Positive integers x and y satisfy the equation $\sqrt{x}+\sqrt{y}=\sqrt{1183}$. What is the minimum possible value of $x+y$?

(A) 585 **(B)** 595 **(C)** 623 **(D)** 700 **(E)** 791

Key idea. Simplify the right side first: $1183 = 7\cdot 13^2$, so $\sqrt{1183} = 13\sqrt{7}$. That suggests $\sqrt{x}$ and $\sqrt{y}$ should both be whole multiples of $\sqrt{7}$, splitting the 13 into two parts.

Step 1. Squaring $\sqrt{x}+\sqrt{y}=\sqrt{1183}$ gives $x+y+2\sqrt{xy}=1183$, so $\sqrt{xy}$ is rational; since xy is an integer, $\sqrt{xy}$ is an integer.

Step 2. Then $\sqrt{x}\cdot(\sqrt{x}+\sqrt{y}) = x+\sqrt{xy}$ is an integer, so $\sqrt{x} = \dfrac{x+\sqrt{xy}}{13\sqrt{7}}$ is a rational multiple of $\sqrt{7}$. Writing $\sqrt{x} = r\sqrt{7}$, we get $x = 7r^2$, and since $7r^2$ is an integer and 7 has no square factor, the denominator of r must be 1: r is a positive integer a. Likewise $\sqrt{y} = b\sqrt{7}$, $y = 7b^2$.

Step 3. Now $a\sqrt{7}+b\sqrt{7} = 13\sqrt{7}$ gives $a+b=13$, and

$$x+y = 7(a^2+b^2).$$

With the sum $a+b$ fixed, a^2+b^2 is smallest when a and b are as close as possible, because $a^2+b^2 = \frac{(a+b)^2+(a-b)^2}{2}$. They cannot be equal (13 is odd), so take $a=6$, $b=7$:

$$x+y = 7(36+49) = 7\cdot 85 = 595.$$

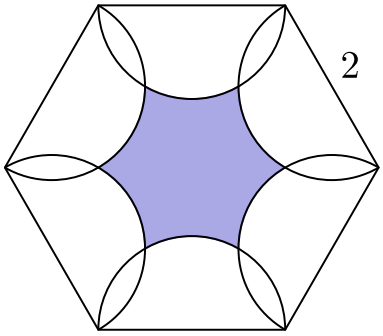
Indeed $x=252$, $y=343$: $\sqrt{252}+\sqrt{343} = 6\sqrt{7}+7\sqrt{7} = 13\sqrt{7}$. For comparison, the splits $1+12$, $2+11$, $3+10$, $4+9$, $5+8$, $6+7$ give 1015, 875, 763, 679, 623, 595 — steadily smaller as the parts even out.

Answer: (B) 595

2 Geometry

Source: 2020 AMC 10B #14

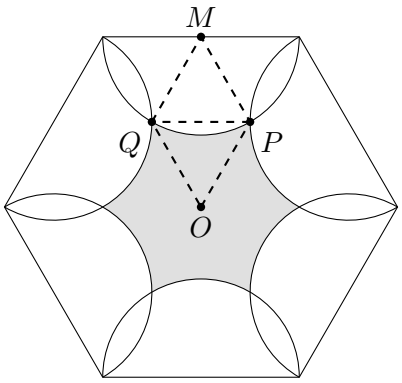
As shown in the figure below, six semicircles lie in the interior of a regular hexagon with side length 2 so that the diameters of the semicircles coincide with the sides of the hexagon. What is the area of the shaded region—inside the hexagon but outside all of the semicircles?



(A) $6\sqrt{3}-3\pi$ **(B)** $\frac{9\sqrt{3}}{2}-2\pi$ **(C)** $\frac{3\sqrt{3}}{2}-\frac{\pi}{3}$ **(D)** $3\sqrt{3}-\pi$ **(E)** $\frac{9\sqrt{3}}{2}-\pi$

Key idea. Subtracting six semicircles from the hexagon double-subtracts their overlaps, so attack the shaded region directly. Its six-fold symmetry says: cut it into six congruent pieces through the center and find one piece.

Step 1: the corners of the shaded region. Let O be the center. Two adjacent semicircles (radius 1) have centers at the midpoints of two sides; both pass through the shared hexagon vertex and cross again at a second point, the mirror image of that vertex in the line joining the two midpoints. The vertex is 2 from O and that line is $\sqrt{3}\cos 30^\circ = \frac{3}{2}$ from O, so the mirror image is $2\cdot\frac{3}{2}-2=1$ from O. Hence the six "star tips" P, Q, … form a regular hexagon of side 1 around O.



Step 2: one piece. Take the top side, with midpoint M. Then $OP=OQ=PQ=1$ (side of the small hexagon) and $MP=MQ=1$ (radii). So $OPMQ$ is a rhombus made of two equilateral triangles of side 1, with area $2\cdot\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$.

The shaded part of this rhombus is what is left after removing the part of the semicircle inside it: a sector centered at M with angle $\angle PMQ=60^\circ$ and radius 1, of area $\frac{60}{360}\cdot\pi\cdot 1^2=\frac{\pi}{6}$.

Step 3: total. One piece has area $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$, so the shaded region has area

$$6\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = 3\sqrt{3} - \pi.$$

Watch out. The hexagon has area $6 \cdot \frac{\sqrt{3}}{4} \cdot 2^2 = 6\sqrt{3}$ and the six semicircles total $6 \cdot \frac{\pi}{2} = 3\pi$, so "hexagon minus semicircles" gives $6\sqrt{3} - 3\pi$, choice (A). But neighboring semicircles overlap in the lens shapes near each vertex, and those lenses get subtracted twice.

Answer: (D) $3\sqrt{3} - \pi$

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3 Counting & Probability

Source: 2021 Fall AMC 10B #16

Five balls are arranged around a circle. Chris chooses two adjacent balls at random and interchanges them. Then Silva does the same, with her choice of adjacent balls to interchange being independent of Chris's. What is the expected number of balls that occupy their original positions after these two successive transpositions?

(A) 1.6 **(B)** 1.8 **(C)** 2.0 **(D)** 2.2 **(E)** 2.4

Key idea. By symmetry it does not matter which pair Chris picks. Fix his pair and sort Silva's 5 equally likely pairs by how they overlap with it.

Step 1. Label the positions $1, \ldots, 5$ around the circle and say Chris swaps the balls in positions 1 and 2. Silva's 5 possible pairs are $\{1,2\}$, $\{2,3\}$, $\{5,1\}$, $\{3,4\}$, $\{4,5\}$, each with probability $\frac{1}{5}$.

Step 2. *Same pair* $\{1,2\}$: she swaps them back, so all 5 balls are home. Probability $\frac{1}{5}$.

Sharing one ball ($\{2,3\}$ or $\{5,1\}$): three balls are cycled around and the other 2 stay put. For example, after $\{1,2\}$ then $\{2,3\}$ the balls originally at $1,2,3$ end at $3,1,2$, while balls 4 and 5 never moved. Probability $\frac{2}{5}$.

Disjoint pair ($\{3,4\}$ or $\{4,5\}$): four balls are displaced and only 1 stays home. Probability $\frac{2}{5}$.

Step 3. The expected number is

$$5 \cdot \frac{1}{5} + 2 \cdot \frac{2}{5} + 1 \cdot \frac{2}{5} = \frac{5+4+2}{5} = \frac{11}{5} = 2.2.$$

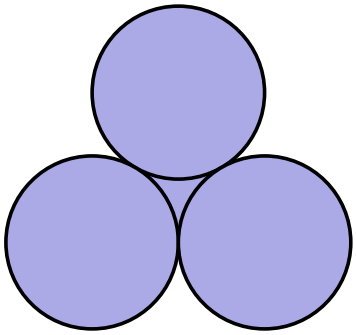
Another way. Use linearity of expectation. A given ball ends at home if neither swap touches it, probability $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$, or if both players pick the same pair containing it, probability $2 \cdot \frac{1}{25} = \frac{2}{25}$. (If only one swap moves it, or two different swaps do, it ends elsewhere.) So each ball is home with probability $\frac{11}{25}$, and the expected count is $5 \cdot \frac{11}{25} = 2.2$.

Answer: (D) 2.2

4 Geometry

Source: 2012 AMC 10B #16

Three circles with radius 2 are mutually tangent. What is the total area of the circles and the region bounded by them, as shown in the figure?



(A) $10\pi + 4\sqrt{3}$ **(B)** $13\pi - \sqrt{3}$ **(C)** $12\pi + \sqrt{3}$ **(D)** $10\pi + 9$ **(E)** 13π

Key idea. The region in question is one solid shape: three disks plus the curved gap between them. Joining the three centers gives a triangle, and the shape is exactly that triangle together with the parts of the disks sticking out of it.

Step 1. Tangent circles of radius 2 have centers $2 + 2 = 4$ apart, so the centers form an equilateral triangle of side 4. Each point of tangency lies on a side of this triangle, so the triangle contains the whole gap. Its area is

$$\frac{\sqrt{3}}{4} \cdot 4^2 = 4\sqrt{3}.$$

Step 2. Each disk is split by the triangle: the part inside is a 60° sector (an angle of the equilateral triangle), and the part outside is a $360^\circ - 60^\circ = 300^\circ$ sector, of area

$$\frac{300}{360} \cdot \pi \cdot 2^2 = \frac{5}{6} \cdot 4\pi = \frac{10\pi}{3}.$$

Step 3. The shape is the triangle plus three 300° sectors:

$$4\sqrt{3} + 3 \cdot \frac{10\pi}{3} = 10\pi + 4\sqrt{3}.$$

Check another way: three full disks have area 12π; the gap is the triangle minus three 60° sectors, $4\sqrt{3} - 3 \cdot \frac{4\pi}{6} = 4\sqrt{3} - 2\pi$. The sum is $12\pi + 4\sqrt{3} - 2\pi = 10\pi + 4\sqrt{3}$.

Watch out. Adding the whole triangle to three *full* disks counts the three 60° sectors twice and gives $12\pi + 4\sqrt{3}$, which is not a choice.

Answer: (A) $10\pi + 4\sqrt{3}$

5 Algebra

Source: 2020 AMC 10A #17

Define

$$P(x) = (x - 1^2)(x - 2^2)\cdots(x - 100^2).$$

How many integers n are there such that $P(n) \le 0$?

(A) 4900 **(B)** 4950 **(C)** 5000 **(D)** 5050 **(E)** 5100

Key idea. A product of factors is negative exactly when an odd number of them are negative. Track how many factors $(n - k^2)$ are negative as n walks past the roots $1^2, 2^2, \ldots, 100^2$.

Step 1: the sign. If $n < 1$, all 100 factors are negative; 100 is even, so $P(n) > 0$. If $n > 100^2$, all factors are positive and $P(n) > 0$.

If $k^2 < n < (k+1)^2$ for some k from 1 to 99, the factors $n - 1^2, \ldots, n - k^2$ are positive and the other $100 - k$ are negative. So $P(n) < 0$ exactly when $100 - k$ is odd, i.e. when k is odd. At the roots themselves $P(n) = 0$.

Step 2: which integers. So $P(n) \le 0$ precisely for the integers in

$$[1^2, 2^2],\ [3^2, 4^2],\ \ldots,\ [99^2, 100^2].$$

Every root k^2 is an endpoint of one of these intervals, and the integers strictly inside the gaps such as $(2^2, 3^2)$ give $P(n) > 0$.

Step 3: count. For odd k, the interval $[k^2, (k+1)^2]$ contains $(k+1)^2 - k^2 + 1 = 2k + 2$ integers. These intervals do not overlap. Summing over $k = 1, 3, \ldots, 99$ (that is 50 values):

$$\sum (2k + 2) = 2(1 + 3 + \cdots + 99) + 2 \cdot 50 = 2 \cdot 2500 + 100 = 5100.$$

Watch out. The number of integers from a to b inclusive is $b - a + 1$, not $b - a$. Dropping the $+1$ counts $2k + 1$ per interval and gives $2 \cdot 2500 + 50 = 5050$, choice (D).

Answer: (E) 5100

6 Geometry

Source: 2023 AMC 10A #19

The line segment formed by $A(1,2)$ and $B(3,3)$ is rotated to the line segment formed by $A'(3,1)$ and $B'(4,3)$ about the point $P(r,s)$. What is $|r-s|$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{2}{3}$ **(E)** 1

Key idea. A rotation moves every point along a circle about the center, so the center is the same distance from a point and its image. That puts P on the perpendicular bisector of $\overline{AA'}$ and on the perpendicular bisector of $\overline{BB'}$.

Step 1. $B(3,3)$ and $B'(4,3)$ lie on a horizontal line, so the perpendicular bisector of $\overline{BB'}$ is the vertical line through the midpoint: $x=\frac{7}{2}$.

Step 2. For $\overline{AA'}$, set $PA^2=PA'^2$:

$$(x-1)^2+(y-2)^2=(x-3)^2+(y-1)^2.$$

Expanding and cancelling x^2 and y^2: $-2x+1-4y+4=-6x+9-2y+1$, so $4x-2y=5$.

Step 3. With $x=\frac{7}{2}$: $14-2y=5$, so $y=\frac{9}{2}$. Thus $P=\left(\frac{7}{2},\frac{9}{2}\right)$.

Check. $PA^2=\left(\frac{5}{2}\right)^2+\left(\frac{5}{2}\right)^2=\frac{25}{2}=PA'^2$ and $PB^2=\left(\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2=\frac{5}{2}=PB'^2$. Both $A\to A'$ and $B\to B'$ turn through the same angle about P (one whose tangent is $\frac{3}{4}$, counterclockwise), so it really is a single rotation.

Therefore $|r-s|=\left|\frac{7}{2}-\frac{9}{2}\right|=1$.

Answer: (E) 1

7 Counting & Probability

Source: 2021 AMC 10A #20

In how many ways can the sequence $1,2,3,4,5$ be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

(A) 10 **(B)** 18 **(C)** 24 **(D)** 32 **(E)** 44

Key idea. "No three in a row going the same way" means the four steps between neighbors must alternate: up, down, up, down or down, up, down, up. Then place the largest number, 5, which can only sit at a peak.

Step 1: the pattern. Write the arrangement as a,b,c,d,e. If two consecutive steps went the same way, those three terms would be increasing or decreasing. So the steps alternate: either $a<b>c<d>e$ or $a>b<c>d<e$.

Step 2: count $a < b > c < d > e$. The terms a, c, e are each smaller than a neighbor, so 5 must be b or d.

Suppose $b = 5$, and let $d = r$. The conditions left are $c < r$ and $e < r$ ($a < 5$ and $c < 5$ hold automatically). So c and e are two different numbers from the $r - 1$ numbers below r, in order: $(r-1)(r-2)$ ways; then a is the last number. For $r = 1, 2, 3, 4$ this gives $0 + 0 + 2 + 6 = 8$.

Mirror images (reverse the arrangement) give another 8 with $d = 5$. So this pattern has 16 arrangements.

Step 3: the other pattern. Replacing every term x by $6 - x$ turns each $a < b > c < d > e$ arrangement into an $a > b < c > d < e$ arrangement and back, so there are also 16 of those.

Total. $16 + 16 = 32$.

Watch out. Counting only one of the two patterns gives 16, which is not a choice — a hint that a pattern was left out.

Answer: (D) 32

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8 Number Theory

Source: 2018 AMC 10B #21

Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, \ldots, \frac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323?

(A) 324 **(B)** 330 **(C)** 340 **(D)** 361 **(E)** 646

Key idea. The size limit $n \le 9999$ is the key: 323^2 is already far above 9999, so the next divisor d cannot be coprime to $323 = 17 \cdot 19$. Sharing a factor with 323 forces d to be at least 17 past it.

Step 1: factor. $323 = 17 \cdot 19$, and $323^2 = 104{,}329$.

Step 2: d must share a factor with 323. Let $d > 323$ be the next divisor. If $\gcd(d, 323) = 1$, then both 323 and d divide n with no common factor, so $323d$ divides n. But $323d > 323 \cdot 323 = 104{,}329 > 9999$, impossible for a 4-digit n. So $g = \gcd(d, 323)$ is 17, 19, or 323.

Step 3: the lower bound. The number g divides both d and 323, so it divides $d - 323$. Since $d - 323 > 0$, we get $d - 323 \ge g \ge 17$, so $d \ge 340$.

Step 4: 340 is achieved. We need n divisible by $323 = 17 \cdot 19$ and $340 = 2^2 \cdot 5 \cdot 17$, i.e. by $\text{lcm} = 2^2 \cdot 5 \cdot 17 \cdot 19 = 6460$. Take $n = 6460$: it is an even 4-digit number, 323 and 340 both divide it, and by Step 3 no divisor lies strictly between them. (Its divisors near there are $\ldots, 190, 323, 340, 380, \ldots$.)

Watch out. Because n is even, $2 \cdot 323 = 646$ always divides n, and for most valid n (such as 1292 or 3230) it really is the next divisor. Stopping there gives 646, choice E — but the question asks for the *smallest possible*.

Watch out. 324 looks smallest, but $\gcd(324, 323) = 1$, so by Step 2 it is impossible; that is choice A.

Answer: (C) 340

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9 Algebra

Source: 2024 AMC 10A #23

Integers a, b, and c satisfy

$$ab + c = 100,$$
$$bc + a = 87,$$
$$ca + b = 60.$$

What is $ab + bc + ca$?

(A) 212 **(B)** 247 **(C)** 258 **(D)** 276 **(E)** 284

Key idea. The equations are nearly symmetric, so subtract two of them: the difference factors. Since the right sides differ by the prime 13 and the variables are integers, the factored form allows only a handful of cases.

Step 1: Subtract and factor. First equation minus second:

$$ab + c - bc - a = 13 \implies b(a - c) - (a - c) = 13 \implies (a - c)(b - 1) = 13.$$

Step 2: The four cases. 13 is prime, so $(a - c,\ b - 1)$ is $(1, 13)$, $(13, 1)$, $(-1, -13)$, or $(-13, -1)$. In each case substitute b and a (in terms of c) into $bc + a = 87$:

- $b = 14$, $a = c + 1$: $14c + c + 1 = 87$, so $15c = 86$. Not an integer.
- $b = 2$, $a = c + 13$: $2c + c + 13 = 87$, so $3c = 74$. Not an integer.
- $b = -12$, $a = c - 1$: $-12c + c - 1 = 87$, so $-11c = 88$, $c = -8$, $a = -9$.
- $b = 0$, $a = c - 13$: $0 + c - 13 = 87$, so $c = 100$, $a = 87$. But then $ca + b = 8700 \neq 60$. Fails.

Step 3: Check and finish. For $(a, b, c) = (-9, -12, -8)$: $ab + c = 108 - 8 = 100$, $bc + a = 96 - 9 = 87$, $ca + b = 72 - 12 = 60$. All three hold. So

$$ab + bc + ca = 108 + 96 + 72 = 276.$$

Another way. Adding all three equations gives $ab + bc + ca + (a + b + c) = 247$. With $a + b + c = -29$, this gives $ab + bc + ca = 247 + 29 = 276$.

Watch out. Adding the equations and stopping at 247, choice (B), forgets that the left side also contains $a + b + c$.

Answer: (D) 276

10 Number Theory

Source: 2017 AMC 10B #23

Let

$$N = 123456789101112\ldots4344$$

be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

(A) 1 **(B)** 4 **(C)** 9 **(D)** 18 **(E)** 44

Key idea. $45 = 5 \times 9$, and both 5 and 9 have quick divisibility tests: the last digit for 5, the digit sum for 9. So find N modulo 5 and modulo 9 separately, then combine.

Step 1. Modulo 5. A number's remainder on division by 5 depends only on its last digit. N ends in 4, so N leaves remainder 4 when divided by 5.

Step 2. Modulo 9. A number leaves the same remainder on division by 9 as its digit sum does. The digits of N are just the digits of $1, 2, \ldots, 44$, so the digit sum of N is the total of the digit sums of $1, 2, \ldots, 44$. Each of these numbers has the same remainder mod 9 as its own digit sum, so

$$N \equiv 1 + 2 + \cdots + 44 = \frac{44 \cdot 45}{2} = 990 \pmod 9.$$

Since $990 = 9 \cdot 110$, N is a multiple of 9.

Step 3. Combine. We need a remainder R with $0 \le R \le 44$ that is a multiple of 9 and leaves remainder 4 when divided by 5. The multiples of 9 in that range are $0, 9, 18, 27, 36$, with remainders $0, 4, 3, 2, 1$ on division by 5. Only $R = 9$ works. (Since 5 and 9 share no common factor, there is exactly one such remainder below 45.)

Watch out. The remainder 4 from Step 1 is the remainder on division by 5, not by 45. Stopping there gives choice (B).

Answer: (C) 9