AMC 10

SET 29

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 13–23 of past AMC 10 contests (2000–2025) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

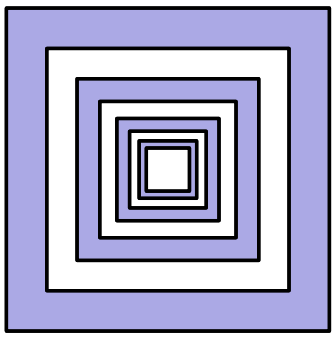
Answer Key and Sources

No.	Ans.	Source	Topic
1	**D**	2025 AMC 10A #13	Algebra
2	**C**	2021 Fall AMC 10B #14	Counting & Probability
3	**A**	2019 AMC 10A #16	Geometry
4	**C**	2021 AMC 10A #16	Algebra
5	**A**	2022 AMC 10A #18	Geometry
6	**A**	2020 AMC 10B #19	Number Theory
7	**B**	2014 AMC 10B #21	Geometry
8	**A**	2025 AMC 10B #22	Counting & Probability
9	**B**	2023 AMC 10B #22	Algebra
10	**D**	2010 AMC 10B #23	Counting & Probability

1 Algebra

Source: 2025 AMC 10A #13

In the figure below, the outside square contains infinitely many squares, each of them with the same center and sides parallel to the outside square. The ratio of the side length of a square to the side length of the next inner square is k, where $0 < k < 1$. The spaces between squares are alternately shaded, as shown in the figure (which is not necessarily drawn to scale).



The area of the shaded portion of the figure is 64% of the area of the original square. What is k?

(A) $\frac{3}{5}$ **(B)** $\frac{16}{25}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{4}{5}$

Key idea. "Infinitely many" shapes shrinking by the same factor each time is the signature of a *geometric series*. Write each shaded ring as a difference of two square areas, add them up, and set the total equal to 64%.

Step 1. Set up the sizes. Since $0 < k < 1$, each square is *smaller* than the one around it by the factor k. Let the outside square have side 1. Then the sides, from the outside in, are $1, k, k^2, k^3, \ldots$, and the areas are $1, k^2, k^4, k^6, \ldots$ — areas shrink by k^2, not k.

Step 2. Add the shaded rings. In the figure the outermost ring is shaded, the next is white, and so on. The first shaded ring has area $1 - k^2$, the next shaded ring has area $k^4 - k^6$, and so on, so

$$\text{shaded area} = (1 - k^2) + (k^4 - k^6) + \cdots = 1 - k^2 + k^4 - k^6 + \cdots .$$

This is a geometric series with first term 1 and ratio $-k^2$, and $0 < k^2 < 1$, so its sum is

$$\frac{1}{1 - (-k^2)} = \frac{1}{1 + k^2}.$$

Step 3. Solve. The original square has area 1, so we need $\dfrac{1}{1 + k^2} = 64\% = \dfrac{16}{25}$. Flipping both sides, $1 + k^2 = \dfrac{25}{16}$, so $k^2 = \dfrac{9}{16}$ and, since $k > 0$, $k = \dfrac{3}{4}$.

Check. With $k = \frac{3}{4}$: $k^2 = \frac{9}{16}$ and $\frac{1}{1+9/16} = \frac{16}{25} = 0.64$. ✓

Watch out. Side lengths shrink by k, but areas shrink by k^2. Using ratio $-k$ instead of $-k^2$ gives $\frac{1}{1+k} = \frac{16}{25}$, so $k = \frac{9}{16}$ — not a choice, which is the hint that the squaring was lost.

Answer: (D) $\frac{3}{4}$

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2 Counting & Probability

Source: 2021 Fall AMC 10B #14

> Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4?
>
> **(A)** $\frac{3}{4}$ **(B)** $\frac{57}{64}$ **(C)** $\frac{59}{64}$ **(D)** $\frac{187}{192}$ **(E)** $\frac{63}{64}$

Key idea. "Divisible by 4" means the product has at least two factors of 2. Failing is much rarer and easier to describe — at most one factor of 2 — so count the complement.

Step 1. Sort the faces by how many factors of 2 they carry: $1, 3, 5$ carry none; 2 and 6 carry exactly one; 4 carries two. Each die is odd with probability $\frac{1}{2}$, shows 2 or 6 with probability $\frac{2}{6} = \frac{1}{3}$, and shows 4 with probability $\frac{1}{6}$.

Step 2. The product is *not* divisible by 4 in exactly two cases.

- No factor of 2: all six dice are odd. Probability $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$.
- Exactly one factor of 2: one die shows 2 or 6 and the other five are odd. There are 6 choices for which die, so the probability is $6 \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^5 = \frac{6}{96} = \frac{4}{64}$.

Every other outcome contains a 4 or at least two dice showing 2 or 6, so it has at least two factors of 2.

Step 3. The complement has probability $\frac{1}{64} + \frac{4}{64} = \frac{5}{64}$, so the answer is

$$1 - \frac{5}{64} = \frac{59}{64}.$$

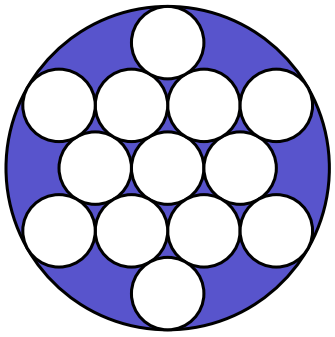
Watch out. Forgetting the "exactly one factor of 2" case leaves only the all-odd case, giving $1 - \frac{1}{64} = \frac{63}{64}$, choice (E). Treating "exactly one even die" as the failing case — letting that die be a 4 as well — gives $1 - \frac{1+6}{64} = \frac{57}{64}$, choice (B). A lone 4 makes the product divisible by 4.

Answer: (C) $\frac{59}{64}$

3 Geometry

Source: 2019 AMC 10A #16

The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1?



(A) $4\pi\sqrt{3}$ **(B)** 7π **(C)** $\pi\left(3\sqrt{3}+2\right)$ **(D)** $10\pi\left(\sqrt{3}-1\right)$ **(E)** $\pi\left(\sqrt{3}+6\right)$

Key idea. Shaded area = big circle − 13 small circles, so everything hinges on the big radius. Tangent unit circles have centers exactly 2 apart, so the centers form a grid of equilateral triangles with side 2 — and distances in such a grid come from the altitude $\sqrt{3}$.

Step 1. The centers. Let O be the center of the middle small circle, which is also the center of the large circle by symmetry. Whenever two small circles are tangent, their centers are $1+1=2$ apart. So the center T of the top circle and the centers of the two circles just below it form an equilateral triangle with side 2, and those two centers together with O form another equilateral triangle with side 2, sharing the same base.

Step 2. The distance OT. An equilateral triangle with side 2 has altitude $\sqrt{2^2-1^2}=\sqrt{3}$. The two triangles sit on opposite sides of their common base, so

$$OT=\sqrt{3}+\sqrt{3}=2\sqrt{3}.$$

The top circle touches the large circle at the point directly above T, so the large radius is

$$R=OT+1=2\sqrt{3}+1.$$

(The four corner circles are also at distance $2\sqrt{3}$ from O, so they touch the large circle too.)

Step 3. The area. Expanding carefully,

$$R^2=(2\sqrt{3}+1)^2=12+4\sqrt{3}+1=13+4\sqrt{3}.$$

The large circle has area $(13+4\sqrt{3})\pi$, and the 13 unit circles have total area 13π, so

$$\text{shaded area}=(13+4\sqrt{3})\pi-13\pi=4\pi\sqrt{3}.$$

Watch out. Do not drop the middle term when squaring: $(2\sqrt{3}+1)^2\neq 12+1$. Without the $4\sqrt{3}$ the shaded area would come out to 0, which is clearly impossible.

Answer: (A) $4\pi\sqrt{3}$

4 Algebra

Source: 2021 AMC 10A #16

In the following list of numbers, the integer n appears n times in the list for $1 \le n \le 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

(A) 100.5 **(B)** 134 **(C)** 142 **(D)** 150.5 **(E)** 167

Key idea. The median is about *positions*, so count how many entries come before each value. Because n appears n times, the last copy of n sits at position $1+2+\cdots+n = \frac{n(n+1)}{2}$, a triangular number.

Step 1. The list has $1 + 2 + \cdots + 200 = \frac{200 \cdot 201}{2} = 20100$ entries. That is even, so the median is the average of the entries in positions 10050 and 10051.

Step 2. Find n with $\frac{n(n+1)}{2}$ close to 10050. Then $n^2 \approx 20100$, and $141^2 = 19881$, $142^2 = 20164$, so try n near 141 or 142:

$$\frac{141 \cdot 142}{2} = 10011, \qquad \frac{142 \cdot 143}{2} = 10153.$$

Step 3. The last 141 is at position 10011 and the last 142 is at position 10153, so positions 10012 through 10153 all hold 142. Both 10050 and 10051 are in that range, so both middle entries are 142 and the median is 142.

Watch out. 100.5 is the median of the *distinct* values $1, 2, \ldots, 200$. Large numbers appear many more times, which pulls the median far above the middle value — choice (A) ignores the repetition.

Answer: (C) 142

5 Geometry

Source: 2022 AMC 10A #18

Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \cdots, T_n$ returns the point $(1, 0)$ back to itself?

(A) 359 **(B)** 360 **(C)** 719 **(D)** 720 **(E)** 721

Key idea. Rotations and reflections about the origin never change the distance from the origin, so track only the *angle* of the point. Doing two consecutive transformations together turns out to be very simple.

Step 1: One transformation. Let the point have polar angle θ (in degrees). Rotating by k° counterclockwise gives angle $\theta + k$. Reflecting across the y-axis sends angle α to $180 - \alpha$. So

$$T_k : \ \theta \longmapsto 180 - \theta - k.$$

Step 2: Two in a row. Applying T_k and then T_{k+1}:

$$\theta \;\mapsto\; 180-\theta-k \;\mapsto\; 180-(180-\theta-k)-(k+1)=\theta-1.$$

So every pair of consecutive transformations simply rotates the point 1° clockwise.

Step 3: Follow $(1,0)$, which starts at angle 0. After an even number $2m$ of steps, the angle is $-m$. That is $0 \pmod{360}$ first when $m=360$, i.e. after 720 steps.

After an odd number $2m+1$ of steps, apply T_{2m+1} to angle $-m$: the angle is $180-(-m)-(2m+1)=179-m$. This is $0 \pmod{360}$ first when $m=179$, i.e. after $2\cdot 179+1=359$ steps.

Step 4. The smaller is 359. (Check: after 358 steps the angle is -179, and T_{359} sends it to $180+179-359=0$.)

Watch out. Seeing that each *pair* rotates by 1°, many students conclude that 360 pairs are needed and answer 720, choice (D). That only checks even step counts; after an odd step the point can land back on $(1,0)$ sooner.

Answer: (A) 359

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6 Number Theory

Source: 2020 AMC 10B #19

In a certain card game, a player is dealt a hand of 10 cards from a deck of 52 distinct cards. The number of distinct (unordered) hands that can be dealt to the player can be written as $158A00A4AA0$. What is the digit A?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 7

Key idea. The number is $\binom{52}{10}$. Computing it exactly is long, but the digit just before the final 0 of $158A00A4AA0$ is A — so we only need a units digit, and units digits can be tracked one factor at a time.

Step 1. Write

$$\binom{52}{10}=\frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot 47\cdot 46\cdot 45\cdot 44\cdot 43}{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}.$$

Cancel: $\frac{52}{4}=13$, $\frac{51}{3}=17$, $\frac{50}{10}=5$, $\frac{49}{7}=7$, $\frac{48}{8\cdot 6}=1$, $\frac{45}{9\cdot 5}=1$, $\frac{44}{2}=22$. What is left is

$$\binom{52}{10}=13\cdot 17\cdot 5\cdot 7\cdot 47\cdot 46\cdot 22\cdot 43=10\cdot\big(13\cdot 17\cdot 7\cdot 47\cdot 46\cdot 11\cdot 43\big),$$

using $5\cdot 22=110=10\cdot 11$.

Step 2. Dividing $158A00A4AA0$ by 10 gives $158A00A4AA$, whose units digit is A. So A is the units digit of $13\cdot 17\cdot 7\cdot 47\cdot 46\cdot 11\cdot 43$.

Step 3. Keep only units digits as we multiply: $3\cdot 7=21\to 1$; $1\cdot 7=7$; $7\cdot 7=49\to 9$; $9\cdot 6=54\to 4$; $4\cdot 1=4$; $4\cdot 3=12\to 2$. So $A=2$.

(Indeed $\binom{52}{10} = 15{,}820{,}024{,}220$.)

Watch out. Do not take the units digit of the whole product: it ends in 0 because of the factor 10, and 0 is not a choice. The digit A is the units digit *after* removing that 10.

Answer: (A) 2

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7 Geometry

Source: 2014 AMC 10B #21

Trapezoid $ABCD$ has parallel sides $\overline{AB}$ of length 33 and $\overline{CD}$ of length 21. The other two sides are of lengths 10 and 14. The angles A and B are acute. What is the length of the shorter diagonal of $ABCD$?

(A) $10\sqrt{6}$ **(B)** 25 **(C)** $8\sqrt{10}$ **(D)** $18\sqrt{2}$ **(E)** 26

Key idea. Drop perpendiculars from D and C to $\overline{AB}$. This splits the trapezoid into a rectangle and two right triangles that share the same unknown height, so two Pythagorean equations pin everything down.

Step 1: Set up. Take $AD = 10$ and $BC = 14$ (swapping the two legs just mirrors the trapezoid, which gives the same two diagonals). Let E and F be the feet of the perpendiculars from D and C to $\overline{AB}$. Since angles A and B are acute, E and F lie on segment $\overline{AB}$, and $EF = CD = 21$.

Let $AE = x$ and let the height be h. Then $FB = 33 - 21 - x = 12 - x$.

Step 2: Two right triangles.

$$x^2 + h^2 = 10^2 = 100, \qquad (12 - x)^2 + h^2 = 14^2 = 196.$$

Subtracting the first from the second: $(12 - x)^2 - x^2 = 96$, i.e. $144 - 24x = 96$, so $x = 2$. Then $h^2 = 100 - 4 = 96$. (Both $AE = 2$ and $FB = 10$ are positive, consistent with the acute angles.)

Step 3: The diagonals. Diagonal AC runs horizontally from A to F, a distance $AE + EF = 2 + 21 = 23$, and rises h:

$$AC = \sqrt{23^2 + 96} = \sqrt{529 + 96} = \sqrt{625} = 25.$$

Diagonal BD runs horizontally from B to E, a distance $33 - 2 = 31$:

$$BD = \sqrt{31^2 + 96} = \sqrt{1057} \approx 32.5.$$

The shorter diagonal is $AC = 25$.

Answer: (B) 25

8 Counting & Probability

Source: 2025 AMC 10B #22

A seven-digit positive integer is chosen at random. What is the probability that the number is divisible by 11, given that the sum of its digits is 61?

(A) $\frac{3}{14}$ **(B)** $\frac{3}{11}$ **(C)** $\frac{2}{7}$ **(D)** $\frac{4}{11}$ **(E)** $\frac{3}{7}$

Key idea. A digit sum of 61 is just 2 short of the maximum $7 \cdot 9 = 63$, so almost every digit is a 9. Count the numbers by where the "missing 2" goes, and use the alternating-sum test for 11.

Step 1: List the numbers. Since $63 - 61 = 2$, all digits are 9 except for a total shortfall of 2. Either one digit is 7 (that digit can be in any of 7 places) or two digits are 8 ($\binom{7}{2} = 21$ choices of places). That is $7 + 21 = 28$ numbers, all equally likely. None has a leading 0, so all are genuine seven-digit numbers.

Step 2: The test for 11. A number is divisible by 11 exactly when (sum of digits in odd positions) − (sum of digits in even positions) is a multiple of 11. Positions $1, 3, 5, 7$ are odd (four digits) and $2, 4, 6$ are even (three digits).

Let d_O and d_E be the shortfalls from 9 in the odd and even positions, so $d_O + d_E = 2$. Then

$$(36 - d_O) - (27 - d_E) = 9 - d_O + d_E = 9 - d_O + (2 - d_O) = 11 - 2d_O.$$

For $d_O = 0, 1, 2$ this is $11, 9, 7$; only $d_O = 0$ gives a multiple of 11.

Step 3: Count. So the whole shortfall of 2 is in the three even positions: one of them is 7 (3 ways) or two of them are 8 ($\binom{3}{2} = 3$ ways), giving 6 numbers. For example $9799999 = 11 \cdot 890909$.

The probability is $\frac{6}{28} = \frac{3}{14}$.

Watch out. Counting only the "two 8s" pattern and forgetting that a single 7 also makes up the shortfall gives $\frac{3}{21} = \frac{1}{7}$ — not a choice, which should send you back to recount.

Answer: (A) $\frac{3}{14}$

9 Algebra

Source: 2023 AMC 10B #22

How many distinct values of x satisfy

$$\lfloor x \rfloor^2 - 3x + 2 = 0,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x?

(A) an infinite number **(B)** 4 **(C)** 2

(D) 3 **(E)** 0

Key idea. The equation mixes x with $\lfloor x \rfloor$. Name the integer $n = \lfloor x \rfloor$: then the equation tells you x exactly in terms of n, and the only thing left is to check that $\lfloor x \rfloor$ really equals n.

Step 1. Let $n = \lfloor x \rfloor$, an integer. The equation gives $3x = n^2 + 2$, so $x = \dfrac{n^2+2}{3}$.

Step 2. This x is a solution exactly when $\lfloor x \rfloor = n$, that is,

$$n \le \frac{n^2+2}{3} < n+1.$$

Multiply by 3. The left inequality is $n^2 - 3n + 2 \ge 0$, i.e. $(n-1)(n-2) \ge 0$. This holds for every integer n, because no integer lies strictly between 1 and 2.

Step 3. The right inequality is $n^2 - 3n - 1 < 0$. The roots of $n^2 - 3n - 1$ are $\frac{3 \pm \sqrt{13}}{2} \approx -0.30$ and 3.30, so the integers that work are $n = 0, 1, 2, 3$.

Step 4. These give $x = \frac{2}{3},\ 1,\ 2,\ \frac{11}{3}$. Check one: $\lfloor \frac{11}{3} \rfloor = 3$, and $9 - 11 + 2 = 0$. ✓ Different n give different x, so there are 4 values.

Watch out. Ignoring the floor and solving $x^2 - 3x + 2 = 0$ gives only $x = 1, 2$, choice C. The solutions $\frac{2}{3}$ and $\frac{11}{3}$ are not integers, so that method can never find them.

Answer: (B) 4

10 Counting & Probability

Source: 2010 AMC 10B #23

> The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?
>
> **(A)** 18 **(B)** 24 **(C)** 36 **(D)** 42 **(E)** 60

Key idea. The center entry is the most constrained: it is larger than three entries and smaller than three others. Fix the center, and the rest of the array falls into place with a few choices.

Write a_{ij} for the entry in row i, column j.

Step 1. The entry 1 must be a_{11}: anywhere else it would have a smaller entry to its left or above it. Likewise 9 must be a_{33}. The center is larger than a_{11}, a_{12}, a_{21} and smaller than a_{23}, a_{32}, a_{33}, so $4 \le a_{22} \le 6$.

Step 2. Center 4. The three entries a_{11}, a_{12}, a_{21} are all less than 4, so they are $1, 2, 3$, with $a_{11} = 1$ and $\{a_{12}, a_{21}\} = \{2, 3\}$ in either order: 2 ways. The digits $5, 6, 7, 8$ fill the pairs $a_{13} < a_{23}$ and $a_{31} < a_{32}$; every other comparison is automatic since these digits are bigger than 4 and smaller than 9. Choosing which two go in the top-right pair gives $\binom{4}{2} = 6$ ways, each with only one increasing order. Total $2 \cdot 6 = 12$.

Step 3. Center 6. Replace every entry k by $10 - k$ and rotate the array 180°. The replacement reverses every inequality and the rotation reverses the rows and columns again, so valid arrays go to valid arrays, and center 4 becomes center 6. This matching works both ways, so there are again 12.

Step 4. Center 5. Choose the three entries $a_{12} < a_{13} < a_{23}$ from $\{2, 3, 4, 6, 7, 8\}$; the other three then fill $a_{21} < a_{31} < a_{32}$ in increasing order. The only remaining conditions are $a_{12} < 5 < a_{23}$ and $a_{21} < 5 < a_{32}$: each chosen triple and its complement must contain a digit below 5 and a digit above 5. This rules out exactly $\{2, 3, 4\}$ and $\{6, 7, 8\}$, leaving $\binom{6}{3} - 2 = 20 - 2 = 18$ arrays.

Step 5. The total is $12 + 18 + 12 = 42$.

Another way. Students who know the *hook length formula* for standard Young tableaux can check this: the hooks of a 3×3 grid are $5, 4, 3$; $4, 3, 2$; $3, 2, 1$, with product 8640, and $\frac{9!}{8640} = \frac{362880}{8640} = 42$.

Answer: (D) 42