AMC 10

SET 30

HALF-LENGTH PRACTICE

SOLUTIONS

Answer key • Sources • Step-by-step solutions

All 10 problems come from positions 13–23 of past AMC 10 contests (2000–2025) and appear in their original wording, in order of their original position. Problems © Mathematical Association of America.

Answer Key and Sources

No.	Ans.	Source	Topic
1	**B**	2008 AMC 10B #13	Algebra
2	**A**	2016 AMC 10A #15	Geometry
3	**E**	2006 AMC 10B #16	Number Theory
4	**D**	2022 AMC 10A #16	Algebra
5	**D**	2021 AMC 10A #17	Geometry
6	**B**	2023 AMC 10B #19	Counting & Probability
7	**E**	2022 AMC 10B #21	Algebra
8	**C**	2025 AMC 10B #21	Counting & Probability
9	**B**	2013 AMC 10A #22	Geometry
10	**C**	2011 AMC 10A #23	Counting & Probability

1 Algebra

Source: 2008 AMC 10B #13

For each positive integer n, the mean of the first n terms of a sequence is n. What is the 2008th term of the sequence?

(A) 2008 **(B)** 4015 **(C)** 4016 **(D)** 4,030,056 **(E)** 4,032,064

Key idea. A mean is a sum divided by a count, so turn the information about means into information about sums. Then one term is the difference of two consecutive sums.

Step 1. The mean of the first n terms is n, so their sum is $n \cdot n = n^2$. Write $S_n = a_1 + a_2 + \cdots + a_n = n^2$.

Step 2. The nth term is what gets added when we go from S_{n-1} to S_n:

$$a_n = S_n - S_{n-1} = n^2 - (n-1)^2 = 2n - 1.$$

(The sequence is $1, 3, 5, 7, \ldots$; for example the first three terms average $\frac{1+3+5}{3} = 3$.)

Step 3. So $a_{2008} = 2 \cdot 2008 - 1 = 4015$.

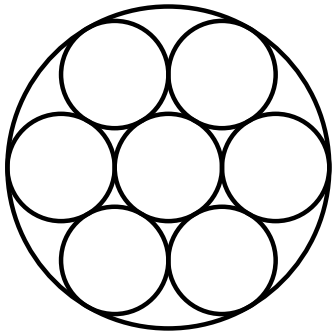
Watch out. $2008^2 = 4{,}032{,}064$ is the *sum* of the first 2008 terms, not the 2008th term — choice (E). And believing the nth term equals n gives (A), which confuses a term with the mean.

Answer: (B) 4015

2 Geometry

Source: 2016 AMC 10A #15

Seven cookies of radius 1 inch are cut from a circle of cookie dough, as shown. Neighboring cookies are tangent, and all except the center cookie are tangent to the edge of the dough. The leftover scrap is reshaped to form another cookie of the same thickness. What is the radius in inches of the scrap cookie?



(A) $\sqrt{2}$ **(B)** 1.5 **(C)** $\sqrt{\pi}$ **(D)** $\sqrt{2\pi}$ **(E)** π

Key idea. Same thickness means volume is proportional to area, so the scrap cookie's area is the dough's area minus the seven cookies' area. First read the dough's radius off the picture.

Step 1. Go from the center of the dough straight out through an outer cookie: you cross the center cookie's radius (1) and then the whole outer cookie (2), reaching the edge. So the dough has radius $1 + 2 = 3$.

Step 2. The dough's area is $\pi \cdot 3^2 = 9\pi$, and the seven cookies take $7 \cdot \pi \cdot 1^2 = 7\pi$. The scrap has area $9\pi - 7\pi = 2\pi$.

Step 3. The scrap cookie has radius r with $\pi r^2 = 2\pi$, so $r^2 = 2$ and $r = \sqrt{2}$.

Watch out. Setting r^2 equal to the scrap area 2π, forgetting to divide by π, gives $\sqrt{2\pi}$, choice D.

Answer: (A) $\sqrt{2}$

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3 Number Theory

Source: 2006 AMC 10B #16

Leap Day, February 29, 2004, occurred on a Sunday. On what day of the week will Leap Day, February 29, 2020, occur?

(A) Tuesday **(B)** Wednesday **(C)** Thursday **(D)** Friday **(E)** Saturday

Key idea. Only the number of days modulo 7 matters. A 365-day year moves the weekday forward 1; every February 29 passed adds one more.

Step 1. From February 29, 2004 to February 29, 2020 is 16 years. Count the days: 16 years of 365 days plus one extra day for each February 29 reached along the way — those of 2008, 2012, 2016 and 2020, which is 4 extra days.

$$16 \cdot 365 + 4 = 5844 \text{ days.}$$

Step 2. Since $365 = 52 \cdot 7 + 1$, each ordinary year contributes 1 day modulo 7. So the weekday moves forward $16 + 4 = 20$ days, and $20 = 2 \cdot 7 + 6$, so effectively 6 days forward.

Step 3. Six days after Sunday is Saturday (equivalently, one day before Sunday).

Watch out. Forgetting that February 29, 2020 itself is one of the leap days crossed gives $16 + 3 = 19 \equiv 5 \pmod 7$, landing on Friday, choice (D). Ignoring leap days altogether gives $16 \equiv 2 \pmod 7$, Tuesday, choice (A).

Answer: (E) Saturday

4 Algebra

Source: 2022 AMC 10A #16

The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

(A) $\frac{24}{5}$ **(B)** $\frac{42}{5}$ **(C)** $\frac{81}{5}$ **(D)** 30 **(E)** 48

Key idea. The new volume is $(h+2)(l+2)(w+2)$: one factor "root $+2$" for each of the three roots. The statement hands you the roots only through the polynomial, and a product over all the roots of $(c - \text{root})$ is exactly what the factored polynomial gives at $x = c$ (times the leading coefficient). Here $c = -2$.

Step 1. Call the roots h, l, w. Since the leading coefficient is 10, the polynomial factors as

$$P(x) = 10x^3 - 39x^2 + 29x - 6 = 10(x - h)(x - l)(x - w).$$

Step 2. Put $x = -2$:

$$P(-2) = 10(-2 - h)(-2 - l)(-2 - w) = -10\,(h + 2)(l + 2)(w + 2),$$

because each of the three factors changes sign, and $(-1)^3 = -1$.

Step 3. Compute $P(-2)$ directly: $10(-8) - 39(4) + 29(-2) - 6 = -80 - 156 - 58 - 6 = -300$.

Step 4. So $-10\,(h + 2)(l + 2)(w + 2) = -300$, and the new volume is $(h + 2)(l + 2)(w + 2) = 30$.

Another way. With Vieta's formulas (dividing by the leading coefficient 10): $hlw = \frac{6}{10} = \frac{3}{5}$, $hl+lw+wh = \frac{29}{10}$, $h + l + w = \frac{39}{10}$. Expanding,

$$(h + 2)(l + 2)(w + 2) = hlw + 2(hl + lw + wh) + 4(h + l + w) + 8 = \tfrac{3}{5} + \tfrac{29}{5} + \tfrac{78}{5} + 8 = \tfrac{110}{5} + 8 = 30.$$

You can even find the box itself: $x = 3$ is a root ($270-351+87-6 = 0$), and $P(x) = (x-3)(2x-1)(5x-2)$, so the box is $3 \times \frac{1}{2} \times \frac{2}{5}$ and the new one is $5 \times \frac{5}{2} \times \frac{12}{5} = 30$.

Watch out. Vieta's formulas for $ax^3 + bx^2 + cx + d$ give $-\frac{b}{a}$, $\frac{c}{a}$, $-\frac{d}{a}$. Forgetting to divide by $a = 10$ gives $6 + 58 + 156 + 8 = 228$, which is not a choice — a sign that something is off.

Answer: (D) 30

5 Geometry

Source: 2021 AMC 10A #17

Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}, BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$.

Given that $OP = 11$, the length of AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

(A) 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215

Key idea. Two features of the statement point the way. $BC = CD$ plus a midpoint P of $\overline{BD}$ means $\overline{CP}$ is perpendicular to $\overline{BD}$ — a second right angle to set beside $\overline{AD} \perp \overline{BD}$. And parallel sides $\overline{AB} \parallel \overline{CD}$ always give similar triangles at the intersection of the diagonals.

Step 1. In isosceles $\triangle BCD$ (apex C), the median $\overline{CP}$ to the base is also an altitude, so $\angle BPC = 90°$.

Step 2. Since $\overline{AB} \parallel \overline{CD}$, alternate angles give $\angle ABD = \angle BDC$. The base angles of $\triangle BCD$ are equal, so $\angle BDC = \angle DBC$. Hence $\angle ABD = \angle PBC$.

Right triangles BPC (right angle at P) and BDA (right angle at D) share this acute angle at B, so $\triangle BPC \sim \triangle BDA$. Matching hypotenuses and legs,

$$\frac{BA}{BC} = \frac{BD}{BP} = 2, \qquad \text{so} \qquad AB = 2 \cdot 43 = 86.$$

Step 3. Again because $\overline{AB} \parallel \overline{CD}$, $\triangle ABO \sim \triangle CDO$, so $\dfrac{BO}{OD} = \dfrac{AB}{CD} = \dfrac{86}{43} = 2$. Thus O is nearer to D than to B, i.e. it lies between P and D.

Write $BP = PD = t$. Then $BO = t + 11$ and $OD = t - 11$, and

$$t + 11 = 2(t - 11) \quad \Longrightarrow \quad t = 33, \qquad BD = 66.$$

(Check: $BO = 44$, $OD = 22$.)

Step 4. In right triangle ABD,

$$AD = \sqrt{86^2 - 66^2} = \sqrt{7396 - 4356} = \sqrt{3040} = \sqrt{16 \cdot 190} = 4\sqrt{190}.$$

Since $190 = 2 \cdot 5 \cdot 19$ has no square factor, $m = 4$, $n = 190$, and $m + n = 194$.

Watch out. Putting O between B and P (so $BO = t - 11$, $OD = t + 11$) contradicts $BO = 2 \cdot OD$. Also simplify fully: $\sqrt{3040} = 2\sqrt{760}$ is not in the required form, because 760 is divisible by 2^2.

Answer: (D) 194

6 Counting & Probability

Source: 2023 AMC 10B #19

Sonya the frog chooses a point uniformly at random lying within the square $[0,6] \times [0,6]$ in the coordinate plane and hops to that point. She then chooses a distance uniformly at random from $[0,1]$ and a direction uniformly at random from {north, south, east, west}. All her choices are independent. She now hops the distance in the chosen direction. What is the probability that she lands outside the square?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{10}$ **(E)** $\frac{1}{9}$

Key idea. There are three random choices, so tame them one at a time. By symmetry the direction does not matter; then *fix* the hop distance d, find the chance of escaping for that d, and finally average over d.

Step 1. Pick a direction. The square looks the same from all four sides, so each direction gives the same escape probability. Hence the answer equals the probability of escaping when she hops *east*. Then only her x-coordinate matters, and it is uniform on $[0,6]$.

Step 2. Fix the distance d. Hopping east by d from x lands at $x+d$, which is outside the square exactly when $x+d>6$, that is, when $x>6-d$. The interval $(6-d,\,6]$ has length d out of the total length 6, so

$$P(\text{escape} \mid d) = \frac{d}{6}.$$

Step 3. Average over d. The distance d is uniform on $[0,1]$, so its average value is $\frac{1}{2}$. The escape probability $\frac{d}{6}$ is just d times the constant $\frac{1}{6}$, so its average is

$$\frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}.$$

(Picture: in the (x,d)-rectangle $[0,6]\times[0,1]$ of area 6, the escape region $x>6-d$ is a right triangle with legs 1 and 1, of area $\frac{1}{2}$; the ratio is $\frac{1/2}{6} = \frac{1}{12}$.)

Watch out. The hop length is random, not always 1. Using $d=1$ for every hop gives $\frac{1}{6}$ — choice (A) — which is the escape chance for the *longest* possible hop, not the average one.

Answer: (B) $\frac{1}{12}$

7 Algebra

Source: 2022 AMC 10B #21

Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial x^2+x+1, the remainder is $x+2$, and when $P(x)$ is divided by the polynomial x^2+1, the remainder is $2x+1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

(A) 10 **(B)** 13 **(C)** 19 **(D)** 20 **(E)** 23

Key idea. Use the first condition to write $P(x) = (x^2+x+1)Q(x)+x+2$, then impose the second condition on Q. The trick that makes the second step easy: when dividing by x^2+1, you may replace x^2 by -1, and then x^2+x+1 simply becomes x.

Step 1. The first condition says

$$P(x) = (x^2+x+1)\,Q(x) + x + 2$$

for some polynomial $Q(x)$ with rational coefficients, and $\deg P = 2 + \deg Q$ when $Q \neq 0$.

Step 2. Find the remainder on division by x^2+1. Since $x^2+x+1 = (x^2+1)+x$, we get

$$P(x) = (x^2+1)\,Q(x) + x\,Q(x) + x + 2,$$

so $P(x)$ and $x\,Q(x)+x+2$ leave the same remainder when divided by x^2+1. We need that remainder to be $2x+1$.

Step 3. Rule out small Q. If $Q=0$, then $P(x)=x+2$, whose remainder on division by x^2+1 is $x+2$, not $2x+1$. If $Q(x)=c$ is a nonzero constant, then $xQ(x)+x+2 = (c+1)x+2$ already has degree below 2, so it is the remainder — but its constant term is 2, not 1. So Q has degree at least 1, and P has degree at least 3.

Step 4. Try $Q(x) = ax+b$. Then

$$xQ(x)+x+2 = ax^2+(b+1)x+2 = a(x^2+1)+(b+1)x+(2-a),$$

so the remainder is $(b+1)x+(2-a)$. Matching with $2x+1$ forces $b=1$ and $a=1$, and only these values work — so the degree-3 polynomial is unique.

Step 5. Expand with $Q(x)=x+1$:

$$P(x) = (x+1)(x^2+x+1)+x+2 = (x^3+2x^2+2x+1)+x+2 = x^3+2x^2+3x+3.$$

The sum of the squares of the coefficients is $1^2+2^2+3^2+3^2 = 1+4+9+9 = 23$.

Check: $x^3+2x^2+3x+3 = (x^2+x+1)(x+1)+(x+2)$ and $x^3+2x^2+3x+3 = (x^2+1)(x+2)+(2x+1)$.

Answer: (E) 23

8 Counting & Probability

Source: 2025 AMC 10B #21

> Each of the 9 squares in a 3×3 grid is to be colored red, blue, or yellow in such a way that each red square shares an edge with at least one blue square, each blue square shares an edge with at least one yellow square, and each yellow square shares an edge with at least one red square. Colorings that can be obtained from one another by rotations and/or reflections are to be considered the same. How many different colorings are possible?
>
> **(A)** 3 **(B)** 9 **(C)** 12 **(D)** 18 **(E)** 27

Key idea. The rule is a cycle: red needs a blue neighbor, blue needs yellow, yellow needs red. The center square is fixed by every rotation and reflection, so split by the center's color; then the edge squares around it are forced almost completely by chasing "who needs whom."

Write R, B, Y for the colors and name the squares

NW	N	NE
W	C	E
SW	S	SE

An edge square touches C and two corners; a corner touches only two edge squares.

Step 1. Rotations and reflections never move C, so colorings with different center colors are never the same. Recoloring every R→B, B→Y, Y→R keeps the rule valid and does not interact with turning the grid, so each center color gives the same number of colorings. Take C = R and multiply by 3 at the end.

Step 2. C needs a blue neighbor, so some edge square is B; rotate it to the top: N = B.

Step 3. No other edge is blue. Suppose E = B. Corner NE then touches only blue squares, so it is R. N needs a yellow neighbor, so NW = Y; NW needs a red neighbor, so W = R. In the same way E forces SE = Y, which forces S = R. Now W needs a blue neighbor, so SW = B — but SW touches only W and S, both red, and has no yellow neighbor. Contradiction (W = B is the mirror image).

Suppose instead S = B. N needs a yellow corner; by the left–right mirror say NW = Y. Then NW forces W = R, W forces SW = B, and SW touches W (red) and S (blue) — no yellow. Contradiction.

Step 4. S is yellow. If S = R, it needs a blue corner, say SE = B (mirror). SE needs yellow, so E = Y. NE now touches N (blue) and E (yellow), so NE ≠ Y, and N's yellow corner must be NW. As before NW forces W = R, W forces SW = B, and SW touches two reds. Contradiction.

Step 5. Finish. Now E, W ∈ {R, Y}.

- E = W = Y: NE and NW each touch only B and Y, so neither can be Y, and N has no yellow neighbor. Impossible.
- E = W = R: NE cannot be B (no yellow neighbor), so E's blue neighbor is SE = B; likewise SW = B. Each of NE, NW is R or Y, not both R (N needs yellow): 3 options, of which (NE, NW) = (Y, R) and (R, Y) are mirror images. 2 *colorings.*

- E = R, W = Y (the case E = Y, W = R is its mirror image): again SE = B. SW touches two yellows, so SW = B. NW touches B and Y, so NW ≠ Y and N forces NE = Y; NW may be R or B. 2 *colorings.*

Each of these 4 grids satisfies every condition (check each square), and the only symmetry other than the identity that keeps the single blue edge N in place is the left–right mirror, already accounted for. So C = R gives 4 colorings, and the total is $3 \cdot 4 = 12$.

Watch out. Do not divide the raw count by 8. There are 84 valid colorings of the fixed grid, and 84/8 is not even an integer: some colorings (for example the left–right symmetric ones) are fixed by a reflection, so their classes have fewer than 8 members.

Answer: (C) 12

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9 Geometry

Source: 2013 AMC 10A #22

Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

(A) $\sqrt{2}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{5}{3}$ **(D)** $\sqrt{3}$ **(E)** 2

Key idea. Turn tangency into distances between centers: externally tangent means the centers are $r_1 + r_2$ apart, internally tangent means $|r_1 - r_2|$ apart. The eighth center is the same distance from all six small centers, so it lies on the line through the hexagon's center perpendicular to its plane — and that line makes a right triangle.

Step 1: the large sphere. A regular hexagon with side 2 has each vertex 2 from its center (it is made of six equilateral triangles of side 2). Let O be that center. Each small sphere has radius 1 and is internally tangent to the large sphere, so the large radius is $2 + 1 = 3$.

Step 2: where the eighth center is. Let the eighth sphere have radius r and center P. It touches all six small spheres externally, so P is at distance $r + 1$ from all six small centers. The points equidistant from the hexagon's vertices lie on the line through O perpendicular to the hexagon's plane. So P is on that line.

Step 3: the two tangencies. Internal tangency with the large sphere gives $OP = 3 - r$. Let Q be any small center: $OQ = 2$, $PQ = r + 1$, and the angle at O is a right angle. By the Pythagorean Theorem,

$$(r + 1)^2 = 2^2 + (3 - r)^2.$$

Expanding, $r^2 + 2r + 1 = 4 + 9 - 6r + r^2$, so $8r = 12$ and $r = \frac{3}{2}$.

Check. With $r = \frac{3}{2}$: $OP = \frac{3}{2}$, and $PQ^2 = 4 + \frac{9}{4} = \frac{25}{4}$, so $PQ = \frac{5}{2} = r + 1$. ✓

Watch out. Using 2 (the hexagon's circumradius) as the large sphere's radius instead of $2+1=3$. That gives $(r+1)^2 = 4+(2-r)^2$, so $r=\frac{7}{6}$ — not a choice.

Answer: (B) $\frac{3}{2}$

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10 Counting & Probability

Source: 2011 AMC 10A #23

Seven students count from 1 to 1000 as follows:

• Alice says all the numbers, except she skips the middle number in each consecutive group of three numbers. That is, Alice says $1, 3, 4, 6, 7, 9, \ldots, 997, 999, 1000$.

• Barbara says all of the numbers that Alice doesn't say, except she also skips the middle number in each consecutive group of three numbers.

• Candice says all of the numbers that neither Alice nor Barbara says, except she also skips the middle number in each consecutive group of three numbers.

• Debbie, Eliza, and Fatima say all of the numbers that none of the students with the first names beginning before theirs in the alphabet say, except each also skips the middle number in each of her consecutive groups of three numbers.

• Finally, George says the only number that no one else says.

What number does George say?

(A) 37 **(B)** 242 **(C)** 365 **(D)** 728 **(E)** 998

Key idea. Track only the numbers that are *skipped* — they are what the next student works with. Alice's skipped numbers $2, 5, 8, \ldots$ form an arithmetic sequence, and skipping the middle of each group of three in an arithmetic sequence produces another arithmetic sequence. So one simple rule, applied six times, finishes the problem.

Step 1 (the rule). Suppose a student's list is the arithmetic sequence a, $a+d$, $a+2d$, ... Her groups of three are $(a, a+d, a+2d)$, $(a+3d, a+4d, a+5d)$, and so on, so the numbers she skips are

$$a+d,\ \ a+4d,\ \ a+7d,\ \ \ldots$$

— a new arithmetic sequence with first term $a+d$ and common difference $3d$. These skipped numbers are exactly the next student's list.

Step 2 (apply it). Alice's list is $1, 2, \ldots, 1000$: first term 1, difference 1.

- Alice skips $2, 5, 8, \ldots, 998$ (first term 2, difference 3; 333 numbers).
- Barbara skips $5, 14, 23, \ldots, 995$ (difference 9; 111 numbers).
- Candice skips $14, 41, 68, \ldots, 986$ (difference 27; 37 numbers).
- Debbie skips $41, 122, 203, \ldots, 932$ (difference 81; 12 numbers).
- Eliza skips $122, 365, 608, 851$ (difference 243; 4 numbers).

Each count comes from the previous one: a list of n numbers has its middles in positions $2, 5, 8, \ldots$, and $1000 \to 333 \to 111 \to 37 \to 12 \to 4$. For example, Debbie's list of 37 has middles in positions $2, 5, \ldots, 35$, which is 12 of them.

Step 3 (the end). Fatima's list is $122, 365, 608, 851$. Her groups are $(122, 365, 608)$ and the leftover (851), so she says $122, 608, 851$ and skips only 365. That is the one number nobody has said, so George says 365.

Another way. The first term goes up by the old difference each time: $1 \to 2 \to 5 \to 14 \to 41 \to 122 \to 365$, adding $1, 3, 9, 27, 81, 243$. After k students the first skipped number is

$$1 + (1 + 3 + \cdots + 3^{k-1}) = 1 + \frac{3^k - 1}{2} = \frac{3^k + 1}{2}.$$

Six students do the skipping, so George's number is $\frac{3^6+1}{2} = \frac{730}{2} = 365$.

Answer: (C) 365